

Existence results for coupled Dirac systems via Rabinowitz-Floer theory

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Abstract

In this paper, we construct the Rabinowitz-Floer homology for the coupled Dirac system

$$\begin{cases} Du = \frac{\partial H}{\partial v}(x, u, v) & \text{on } M, \\ Dv = \frac{\partial H}{\partial u}(x, u, v) & \text{on } M, \end{cases}$$

where M is an n -dimensional compact Riemannian spin manifold, D is the Dirac operator on M , and $H : \Sigma M \oplus \Sigma M \rightarrow \mathbb{R}$ is a real valued superquadratic function of class C^1 with subcritical growth rates. Solutions of this system can be obtained from the critical points of a Rabinowitz-Floer functional on a product space of suitable fractional Sobolev spaces. In particular, we consider the S^1 -equivariant H that includes a nonlinearity of the form

$$H(x, u, v) = f(x) \frac{|u|^{p+1}}{p+1} + g(x) \frac{|v|^{q+1}}{q+1},$$

where $f(x)$ and $g(x)$ are strictly positive continuous functions on M , and $p > 1, q > 1$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-1}{n}.$$

We establish the existence of a nontrivial solution by computing the Rabinowitz-Floer homology in the Morse-Bott situation.

Keywords. Coupled Dirac system; Rabinowitz-Floer homology; Strongly indefinite functionals

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1 Introduction and main results

Let (M, g) be an n -dimensional compact oriented Riemannian manifold equipped with a spin structure $\rho : P_{\text{Spin}(M)} \rightarrow P_{\text{SO}(M)}$, and let $\Sigma M = \Sigma(M, g) = P_{\text{Spin}(M)} \times_{\sigma} \Sigma_n$ denote the complex spinor bundle on M . The latter is a complex vector bundle of rank $2^{\lfloor n/2 \rfloor}$ endowed with the spinorial Levi-Civita connection ∇ and a pointwise Hermitian scalar product. In the following, let $\langle \cdot, \cdot \rangle$ always denote the real part of the Hermitian product on ΣM . It induces a natural inner product $(u, v)_{L^2} = \int_M \langle u(x), v(x) \rangle dx$ on the space $C^\infty(M, \Sigma M)$ of all C^∞ -sections of the bundle ΣM , where dx is the Riemannian measure of g . Denote by $L^2(M, \Sigma M)$ the completion Hilbert space of $C^\infty(M, \Sigma M)$. The Dirac operator is an elliptic differential operator of order one, $D = D_g : C^\infty(M, \Sigma M) \rightarrow C^\infty(M, \Sigma M)$, locally given by $D\psi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi$ for $\psi \in C^\infty(M, \Sigma M)$ and a local g -orthonormal frame $\{e_i\}_{i=1}^n$ of the tangent bundle TM . Consider Whitney direct sum $\Sigma M \oplus \Sigma M$ of ΣM and itself, and write a point of it as (x, ξ, ζ) , where $x \in M$ and $\xi, \zeta \in \Sigma_x M$. Nonlinear Dirac equations arise in many interesting problems in geometry and physics including Dirac-harmonic maps describing the generalized Weierstrass representation of surfaces in three-manifolds [24] and the supersymmetric nonlinear sigma model in quantum field theory [14, 15, 16]. In this paper we will construct the Rabinowitz-Floer homology to study the following system of the coupled semilinear Dirac equations:

$$\begin{cases} Du = \frac{\partial H}{\partial v}(x, u, v) & \text{on } M, \\ Dv = \frac{\partial H}{\partial u}(x, u, v) & \text{on } M, \end{cases} \quad (1.1)$$

where $u, v \in C^1(M, \Sigma M)$ are spinors and $H : \Sigma M \oplus \Sigma M \rightarrow \mathbb{R}$ is a continuous function. (1.1) is the Euler-Lagrange equation of the functional

$$\mathcal{L}_H(u, v) = \int_M (\langle Du, v \rangle - H(x, u, v)) dx. \quad (1.2)$$

The functional \mathcal{L}_H is strongly indefinite since the spectrum of the operator D is unbounded from below and above.

The problem (1.1) can be viewed as a spinorial analogue of other strongly indefinite variational problems such as infinite dynamical systems [9, 10] and elliptic systems [7, 19], and in quantum physics it describes a coupled fermionic fields, and this is our main motivation for its study. A typical way to deal with such problems is the min-max method of Benci and Rabinowitz [11], including the mountain pass theorem, linking arguments and so on. For example, Isobe [28, 29] and authors [23] used this method to study the existence of solutions of generalized nonlinear Dirac equations $Du = H_u(x, u)$ on a compact oriented spin Riemannian manifold. Another way to solve them is homological approach by using Morse theory or Floer homology as in [2, 8, 30, 33]. Inspired by the Rabinowitz-Floer homological method in [4, 5, 17, 18], Maalaoui [35] studied the existence of solutions of the following subcritical Dirac equation

$$Du = |u|^{p-1}u \quad \text{on } M, \quad (1.3)$$

where $1 < p < \frac{n+1}{n-1}$, by constructing Rabinowitz-Floer homology. Recently, he also extended his results to a class of non-linear problems with the so-called starshaped potential [36]. Comparing these two methods, it seems that the homological approach is more “intrinsic” in the sense that the topology of the space of solutions is invariant under perturbations of the subcritical exponent p .

In the following we assume that two real numbers $p, q > 1$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-1}{n}. \quad (1.4)$$

It is not hard to verify that we can choose a real number $s \in (0, 1)$ such that

$$p < \frac{n+2s}{n-2s} \quad \text{and} \quad q < \frac{n+2-2s}{n+2s-2}. \quad (1.5)$$

On nonlinearity H , we make the following hypotheses:

(H0) $H \in C^0(\Sigma M \oplus \Sigma M, \mathbb{R})$ is C^1 in the fiber direction, and C^2 in the fiber direction of $\Sigma M \oplus \Sigma M \setminus \{0\}$.

(H1) There exists a constant $c_0 \in (0, 2)$ such that

$$\langle H_u(x, u, v), u \rangle + \langle H_v(x, u, v), v \rangle \geq 2H(x, u, v) - c_0 \quad (1.6)$$

for all (x, u, v) .

(H2) There exists a constant $c_1 > 0$ such that

$$|H_u(x, u, v)| \leq c_1 \left(1 + |u|^p + |v|^{\frac{p(q+1)}{p+1}} \right), \quad (1.7)$$

$$|H_v(x, u, v)| \leq c_1 \left(1 + |u|^{\frac{q(p+1)}{q+1}} + |v|^q \right). \quad (1.8)$$

(H3) There exist constants $\delta > 0$ and $c_2 > 0$ such that for $|z| > \delta$ with $z = (u, v)$,

$$|H_{uu}(x, u, v)| \leq c_2(1 + |u|^{p-1}), \quad (1.9)$$

$$|H_{vv}(x, u, v)| \leq c_2(1 + |v|^{q-1}), \quad (1.10)$$

$$|H_{uv}(x, u, v)| \leq c_2, \quad |H_{vu}(x, u, v)| \leq c_2. \quad (1.11)$$

(H4) For any $a \in \mathbb{R}$, the map

$$\mathcal{T} : E_s \rightarrow L^{\frac{2n}{n+2s}}(M, \Sigma M) \times L^{\frac{2n}{n+2(1-s)}}(M, \Sigma M)$$

given by $\mathcal{T}(z) = (H_u(x, z), H_v(x, z))^T$, is bounded on the set

$$\Sigma_a(H) = \left\{ z \in E_s \mid \int_M H(x, z(x)) dx \leq a \right\},$$

where $E_s = H^s(M, \Sigma M) \times H^{1-s}(M, \Sigma M)$, see (2.4) for its definition.

Note: Since the equation (1.1) and the following assumptions **(H2)**-**(H4)** are invariant after adding a constant to H , the assumption that $c_0 < 2$ in **(H1)** is unnecessary. We assume it so that the proof of Proposition 3.1 becomes simple.

Consider the following typical examples satisfying the above **(H0)** – **(H4)**,

$$H(x, u, v) = f(x) \frac{|u|^{p+1}}{p+1} + g(x) \frac{|v|^{q+1}}{q+1}, \quad (1.12)$$

where $f(x)$ and $g(x)$ are strictly positive continuous functions on M . Then (1.1) reduces to the following form

$$\begin{cases} Du = g(x)|v|^{p-1}v & \text{on } M, \\ Dv = f(x)|u|^{q-1}u & \text{on } M. \end{cases} \quad (1.13)$$

Note that $\int_M H(x, u, v)dx$ is not well-defined on the Hilbert space $H^{\frac{1}{2}}(M, \Sigma M) \times H^{\frac{1}{2}}(M, \Sigma M)$ unless we make a stronger hypothesis on the exponents p, q as in [19]. The analytic framework in [36] did not work well for our problem and so Maalaoui and Martino's result cannot directly lead to the existence of the solutions of (1.13). To overcome this difficulty, inspired by the ideas of Hulshof and Van der Vorst [27], we consider the following well-defined functional

$$\mathcal{A}_H(u, v, \lambda) = \int_M \langle Du, v \rangle dx - \lambda \int_M (H(x, u, v) - 1) dx \quad (1.14)$$

on a fractional Sobolev space $H^s(M, \Sigma M) \times H^{1-s}(M, \Sigma M) \times \mathbb{R}$ as an analogue of one used by Rabinowitz [37]. From § 2 to § 6 we shall construct, under the suitable assumptions on H , the Rabinowitz-Floer homology in Morse and Morse-Bott situations, respectively. For the latter case, that is, the critical manifold consists of connected components with different dimensions, in contrast to breaking the symmetry via a small perturbation to construct the S^1 -equivalent homology as in [35], we shall follow [12, 13] construct the Morse-Bott homology as follows: choose a Morse function on the critical manifold and define the chain complex to be the \mathbb{Z}_2 -vector space generated by the critical points of this Morse function, while the boundary operator is defined by counting flow lines with cascades. The advantage of this method is that there exists a nice grading for such a complex and the Rabinowitz-Floer homology for $H_0(x, u, v) = \frac{1}{2}(|u|^2 + |v|^2)$ can be partially worked out in Section 7. Based on these we prove the following result.

Theorem 1.1. *Assume that $n \geq 2$ and $0 \notin \text{Spec}(D)$. Problem (1.13) has at least a nontrivial solution $(u, v) \in C^1(M, \Sigma M) \times C^1(M, \Sigma M)$.*

The same method can also be used to derive analogue existence results for a larger class of homogeneous nonlinearities H . Of course, if $H \in C^2(\Sigma M \oplus \Sigma M)$ satisfies **(H1)** – **(H4)** then the functional \mathcal{L}_H in (1.2) is of class C^2 by Proposition 2.1. The methods in [28] can be used to prove some results on existence and multiplicity for solutions of (1.1) under certain further assumptions on H . We can use the saddle point reduction to study it as done in [43] for Dirac equations. These will be given in other places.

Organization of the paper. In section 2, we define a Rabinowitz-Floer functional on a suitable product space of fractional Sobolev spaces, and the perturbed gradient flow. The aim of Section 3 is to prove the $(PS)_c$ condition and boundedness of the perturbed flows. In section 4, we define

and study the relative index and moduli space of trajectories. Section 5 constructs the Rabinowitz Floer homology in Morse and Morse-Bott situations, and also proves continuation invariance of the homology. In section 6, we establish the transversality result. Finally, we compute the Rabinowitz-Floer homology and prove Theorem 1.1 in section 7.

2 The analytic framework

Let (M, g) be as in Section 1. The Dirac operator $D = D_g : C^\infty(M, \Sigma M) \rightarrow C^\infty(M, \Sigma M)$ is essentially self-adjoint in $L^2(M, \Sigma M)$ and its spectrum consists of an unbounded sequence of real numbers (cf. [24, 34]). The well known Schrödinger-Lichnerowicz formula implies that all eigenvalues of D are nonzero if M has positive scalar curvature. Hereafter, *we assume*:

$$0 \notin \text{spec}(D) \quad \text{and} \quad \int_M dx = 1 \quad \text{i.e., the volume of } (M, g) \text{ equals to 1.}$$

(The second assumption is only for simplicity, it is actually unnecessary for our result!).

Let $(\psi_k)_{k=1}^\infty$ be a complete L^2 - orthonormal basis of eigenspinors corresponding to the eigenvalues $(\lambda_k)_{k=1}^\infty$ counted with multiplicity such that $|\lambda_k| \rightarrow \infty$ as $k \rightarrow \infty$. For each $s \geq 0$, let $H^s(M, \Sigma M)$ be the Sobolev space of fractional order s , its dual space is denoted by $H^{-s}(M, \Sigma M)$. We have a linear operator $|D|^s : H^s(M, \Sigma M) \subset L^2(M, \Sigma M) \rightarrow L^2(M, \Sigma M)$ defined by

$$|D|^s u = \sum_{k=1}^\infty a_k |\lambda_k|^s \psi_k, \quad (2.1)$$

where $u = \sum_{k=1}^\infty a_k \psi_k \in H^s(M, \Sigma M)$. Since $0 \notin \text{spec}(D)$ the inverse $|D|^{-s} \in \mathcal{L}(L^2(M, \Sigma M))$ is compact and self-adjoint. $|D|^s$ can be used to define a new inner product on $H^s(M, \Sigma M)$,

$$(u, v)_{s,2} := (|D|^s u, |D|^s v)_2. \quad (2.2)$$

The induced norm $\|\cdot\|_{s,2} = \sqrt{(\cdot, \cdot)_{s,2}}$ is equivalent to the usual one on $H^s(M, \Sigma M)$ (cf. [1, 6]).

For $r \in \mathbb{R}$ consider the Hilbert space

$$\bar{\omega}^{2r} = \left\{ \mathbf{a} = (a_1, a_2, \dots) \mid \sum_{k=1}^\infty a_k^2 \lambda_k^{2r} < \infty \right\}$$

with inner product

$$\langle \mathbf{a}, \mathbf{b} \rangle_{2r} = \sum_{k=1}^\infty \lambda_k^{2r} a_k b_k.$$

Then $H^s(M, \Sigma M)$ can be identified with the Hilbert space $\bar{\omega}^{2s}$. Hence

$$H^{-s}(M, \Sigma M) = (H^s(M, \Sigma M))'$$

can be identified with $\bar{\omega}^{-2s}$, where the pairing between $\bar{\omega}^{-2s}$ and $\bar{\omega}^{2s}$ is given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{k=1}^\infty a_k b_k.$$

It follows that $|D|^{-2s}$ gives a Hilbert space isomorphism from $H^{-s}(M, \Sigma M)$ to $H^s(M, \Sigma M)$ with respect to the equivalent new inner products as in (2.2). Moreover we have a continuous inclusion $L^2(M, \Sigma M) \hookrightarrow H^{-s}(M, \Sigma M)$ and

$$(|D|^{-2s}u, v)_{s,2} := (u, v)_2 \quad \forall u, v \in L^2(M, \Sigma M). \quad (2.3)$$

Consider the Hilbert space

$$E_s := H^s(M, \Sigma M) \times H^{1-s}(M, \Sigma M) \quad (2.4)$$

with norm $\|z\| := (\|u\|_s^2 + \|v\|_{1-s}^2)^{\frac{1}{2}}$ for $z = (u, v) \in E_s$. By the Sobolev embedding theorem, we have the compact embedding $E_s \hookrightarrow L^{p+1}(M, \Sigma M) \times L^{q+1}(M, \Sigma M)$. Let $E_s^* = H^{-s}(M, \Sigma M) \times H^{-(1-s)}(M, \Sigma M)$, which is the dual space of E_s . Then

$$\mathcal{D}_s := \begin{pmatrix} |D|^{-2s} & 0 \\ 0 & |D|^{-2(1-s)} \end{pmatrix} : E_s^* \rightarrow E_s$$

is a Hilbert space isomorphism by the arguments above (2.3) and

$$(\mathcal{D}_s z_1, z_2)_{E_s} = (z_1, z_2)_{L^2} \quad (2.5)$$

for any $z_1, z_2 \in L^2(M, \Sigma M) \times L^2(M, \Sigma M)$.

Since M is compact, by the assumption **(H1)** we have constants $C_1, C_2 > 0$ such that

$$|H(x, u, v)| \geq C_1(|u|^2 + |v|^2) - C_2 \quad \forall (x, u, v), \quad (2.6)$$

and by the assumption **(H2)** we can use Young's inequality to derive

$$|H(x, u, v)| \leq C(1 + |u|^{p+1} + |v|^{q+1}) \quad \forall (x, u, v) \quad (2.7)$$

for some constant $C > 0$. (Later on, we also use C to denote various positive constants independent of u and v without special statements). (2.6) and (2.7) show that the nonlinearity H is asymptotically quadric or superquadric.

From now on we also assume that

$$H \in C^2(\Sigma M \oplus \Sigma M). \quad (2.8)$$

Proposition 2.1. *Assume that $H \in C^1(\Sigma M \oplus \Sigma M)$ satisfies **(H1)** – **(H2)** and **(H4)**. Then the functional $\mathcal{H} : E_s \rightarrow \mathbb{R}$ defined by*

$$\mathcal{H}(x, u, v) = \int_M H(x, u(x), v(x)) dx, \quad (2.9)$$

is of class C^1 , its derivation at $(u, v) \in E_s$ is given by

$$\mathcal{H}'(u, v)(\xi, \zeta) = \int_M (\langle H_u(x, u, v), \xi \rangle + \langle H_v(x, u, v), \zeta \rangle) dx \quad \forall (\xi, \zeta) \in E_s, \quad (2.10)$$

and $\mathcal{H}' : E_s \rightarrow E_s^* \equiv E_s$ is a compact map. Furthermore, if this H also belongs to $C^2(\Sigma M \oplus \Sigma M)$ and satisfies **(H3)**, then \mathcal{H} is of class C^2 .

Remark 2.2. If the real numbers p, q satisfy

$$1 < p, q < \min \left\{ \frac{n+2s}{n-2s}, \frac{n+2(1-s)}{n-2(1-s)} \right\}$$

for some $s \in (0, 1)$, which implies (1.4), the above space E_s can be replaced by $E_{\frac{1}{2}}$. In particular, for $n = \dim M = 2$ and $2 < p, q < 3$, we can prove that the functional $\mathcal{H} : E_{\frac{1}{2}} \rightarrow \mathbb{R}$ is of class C^3 provided that $H \in C^3(\Sigma M \oplus \Sigma M)$ satisfies (H1) – (H4), and that suitable growth conditions on $H_{uuu}, H_{vvv}, H_{uuv}$ and H_{uvv} are applied. Of course, for $n = \dim M = 1$ it can also be proved that the functional \mathcal{H} is of class C^∞ on $E_{\frac{1}{2}}$ if $H \in C^\infty(\Sigma M \oplus \Sigma M)$ satisfies suitable conditions.

For the sake of completeness we shall give the proof of Proposition 2.1 in Appendix A.

It follows that the Rabinowitz-Floer functional \mathcal{A}_H in (1.14) is of class C^2 on Hilbert space $\mathcal{E} := E_s \times \mathbb{R}$ with inner product

$$((\xi_1, \mu_1), (\xi_2, \mu_2))_{\mathcal{E}} = (\xi_1, \xi_2)_{E_s} + \mu_1 \cdot \mu_2 \quad (2.11)$$

for $(\xi_i, \mu_i) \in \mathcal{E}, i = 1, 2$. Moreover, $(u, v, \lambda) \in \mathcal{E}$ is a critical point of \mathcal{A}_H if and only if

$$\begin{cases} Du = \lambda H_v(x, u, v) & \text{on } M, \\ Dv = \lambda H_u(x, u, v) & \text{on } M, \\ \int_M H(x, u, v) dx = 1 & \text{on } M. \end{cases} \quad (2.12)$$

Since $\langle Du, v \rangle = \langle u, Dv \rangle$ and $\int_M \langle Du, v \rangle dx = (Du, v)_2$, the functional \mathcal{A}_H can be written as

$$\mathcal{A}_H(z, \lambda) = \frac{1}{2} \int_M \langle Lz(x), z(x) \rangle dx - \lambda \int_M (H(x, z(x)) - 1) dx, \quad (2.13)$$

where

$$L = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}.$$

Note that $\int_M \langle Lz(x), z(x) \rangle dx = (Lz, z)_2 = (\mathcal{D}_s Lz, z)_{E_s}$ by (2.5). We deduce that the gradient of \mathcal{A}_H with respect to the metric (2.11) is given by

$$\nabla \mathcal{A}_H(z, \lambda) = \begin{pmatrix} \mathcal{D}_s \{Lz - \lambda H_z(x, z)\} \\ - \int_M (H(x, z) - 1) dx \end{pmatrix}, \quad (2.14)$$

where $H_z(x, z) = (H_u(x, u, v), H_v(x, u, v))^T$. Proposition 2.1 implies that $\nabla \mathcal{A}_H$ is of class C^1 on E_s . Hence the following system of PDE's

$$\begin{cases} \frac{\partial z}{\partial t} = -\mathcal{D}_s \{Lz - \lambda H_z(x, z)\}, \\ \frac{\partial \lambda}{\partial t} = \int_M (H(x, z) - 1) dx. \end{cases} \quad (2.15)$$

has a local flow on \mathcal{E} . But the initial value problem for the L^2 - gradient flow is ill-posed since the spectrum of D is unbounded from below. We work on E_s which makes the absence of the symmetry of u and v by imposing more regularity of u than of v if p is large and q is small, and vice versa.

The perturbed flows. To obtain transversality, We shall follow the idea of Angenent and Vorst [7] to perturb the metric on $\mathcal{E} = E_s \oplus \mathbb{R}$ and thus make all connecting orbits between critical points to be transverse.

Let $C^2 = C^2(M, \Sigma M \oplus \Sigma M)$, which is a separable Banach space; see [32]. By the definition (cf. [25]), a nuclear operator T from \mathcal{E} to $C^2 \oplus \mathbb{R}$ is a bounded linear operator which can be written as an absolutely convergent sum $\sum_{k=1}^{\infty} y_k \otimes x_k^*$, where $x_k^* \in \mathcal{E}^*$ and $y_k \in C^2 \oplus \mathbb{R}$. The norm of T is defined by

$$\|T\|_{\mathcal{NS}} = \inf \sum_{k=1}^{\infty} \|x_k^*\|_{\mathcal{E}^*} \|y_k\|_{C^2 \oplus \mathbb{R}}.$$

Consider the space

$$\mathcal{NS}(\mathcal{E}, C^2 \oplus \mathbb{R}) := \left\{ K \in \mathcal{L}(\mathcal{E}, C^2 \oplus \mathbb{R}) \left| \begin{array}{l} K \text{ is nuclear and symmetric with} \\ \text{respect to the inner product of } \mathcal{E} \end{array} \right. \right\}.$$

It is a separable Banach space with respect to the above norm $\|\cdot\|_{\mathcal{NS}}$, and contains the space of finite rank operator from \mathcal{E} to $C^2 \oplus \mathbb{R}$ as a dense subspace.

Let $K : \mathcal{E} \rightarrow \mathcal{NS}(\mathcal{E}, C^2 \oplus \mathbb{R})$ be a smooth map of form

$$K(w) = e^{-\|w\|_{\mathcal{E}}^2} \tilde{K}(w), \quad (2.16)$$

where $\tilde{K} \in C^\infty(\mathcal{E}, \mathcal{NS}(\mathcal{E}, C^2 \oplus \mathbb{R}))$ satisfies the Gevrey type estimates

$$\sup_{n \geq 0} \frac{\sup_{w \in \mathcal{E}} \|\tilde{K}^{(n)}(w)\|_{\mathcal{L}_n(\mathcal{E}, \mathcal{NS}(\mathcal{E}, C^2 \oplus \mathbb{R}))}}{(n!)^2} < \infty. \quad (2.17)$$

Denote by \mathbf{K}_0 the set of such maps K such that

$$\sup_{w \in \mathcal{E}} \|K(w)\|_{\mathcal{L}(\mathcal{E}, \mathcal{E})} < \frac{1}{2}. \quad (2.18)$$

(Note: for any $w \in \mathcal{E}$, $K(w) \in \mathcal{NS}(\mathcal{E}, C^2 \oplus \mathbb{R}) \subset \mathcal{L}(\mathcal{E}, C^2 \oplus \mathbb{R})$ and $C^2 \oplus \mathbb{R} \hookrightarrow \mathcal{E}$ is continuous, so $K(w) \in \mathcal{L}(\mathcal{E}, \mathcal{E})$.) The norm of K is defined to be the left side of inequality in (2.17). Then \mathbf{K}_0 is a Banach space with respect to this norm. Note that the space \mathbf{K}_0 contains maps of the form

$$\rho(\|w - w_0\|)k_0, \quad (2.19)$$

where $k_0 \in \mathcal{NS}(\mathcal{E}, C^2 \oplus \mathbb{R})$ is a constant, and $\rho(t) = e^{-1/(1-t^2)}$ for $t < 1$, and $\rho(t) = 0$ for $t \geq 1$. We define a closed linear subspace of \mathbf{K}_0 ,

$$\mathbf{K} = \text{span}(\{\text{all maps of form (2.19)}\}). \quad (2.20)$$

Each $K \in \mathbf{K}$ can yield a perturbed Riemannian metric g^K on \mathcal{E} defined by

$$g_w^K(\xi_1, \xi_2) = (\xi_1, (I + K(w))^{-1} \xi_2)_{\mathcal{E}}, \quad (2.21)$$

where $\xi_i \in T_w \mathcal{E} = \mathcal{E}$, $i = 1, 2$. Then the gradient of \mathcal{A}_H with respect to g^K is given by

$$\nabla^K \mathcal{A}_H(w) = (I + K(w)) \nabla \mathcal{A}_H(w), \quad (2.22)$$

and the modified gradient flow becomes

$$\frac{dw(t)}{dt} + \nabla^K \mathcal{A}_H(w(t)) = 0. \quad (2.23)$$

Denote by Pr_1 the projection from \mathcal{E} to E_s . From (2.16) we get

$$\|\text{Pr}_1(K(w) \nabla \mathcal{A}_H(w))\|_{C^2} \leq C. \quad (2.24)$$

Proposition 2.3. *For any $x \in \mathcal{E} \setminus \{0\}$ and $y \in C^2 \oplus \mathbb{R}$, there exists a $K \in \mathcal{NS}(\mathcal{E}, C^2 \oplus \mathbb{R})$ satisfying $K(x) = y$.*

Proof. As noted above the space of finite rank operators from \mathcal{E} to $C^2 \oplus \mathbb{R}$ is dense in the space of nuclear operators. If $(x, y)_\mathcal{E} = 0$, by choosing $\xi \in C^2 \oplus \mathbb{R}$ with $(x, \xi)_\mathcal{E} \neq 0$ we define

$$K(w) = \frac{(w, \xi)_\mathcal{E}}{(x, \xi)_\mathcal{E}} y + \frac{(w, y)_\mathcal{E}}{(x, \xi)_\mathcal{E}} \xi, \quad \forall w \in \mathcal{E}. \quad (2.25)$$

If $(x, y)_\mathcal{E} \neq 0$, we put

$$K(w) = \frac{(w, y)_\mathcal{E}}{(x, y)_\mathcal{E}} y, \quad \forall w \in \mathcal{E}. \quad (2.26)$$

In both cases, K is a finite rank operator which is symmetric with respect to the inner product in (2.11). \square

3 $(PS)_c$ condition and boundedness of the perturbed flows

In this section we always assume that $H \in C^2(\Sigma M \oplus \Sigma M)$ satisfies (H1) – (H4) without special statements.

3.1 $(PS)_c$ condition

Proposition 3.1. *Suppose that $H \in C^1(\Sigma M \oplus \Sigma M)$ satisfies (H1) – (H2) and (H4). Then the functional \mathcal{A}_H satisfies the $(PS)_c$ condition; that is, suppose that a sequence $\{(z_k, \lambda_k)\}_{k=1}^\infty \subset \mathcal{E}$ satisfies $\mathcal{A}_H(z_k, \lambda_k) \rightarrow c \in \mathbb{R}$ and*

$$\|\nabla \mathcal{A}_H(z_k, \lambda_k)\|_\mathcal{E} = \|d\mathcal{A}_H(z_k, \lambda_k)\|_{\mathcal{E}^*} \rightarrow 0 \text{ as } k \rightarrow \infty,$$

then it has a convergent subsequence.

Proof. Since $\|\nabla \mathcal{A}_H(z_k, \lambda_k)\|_\mathcal{E}^2 = \|\mathcal{D}_s\{Lz_k - \lambda_k H_z(x, z_k)\}\|_{E_s}^2 + \left(\int_M (H(x, z_k) - 1) dx\right)^2$ by (2.14), we have

$$\begin{cases} \epsilon_k := \|\mathcal{D}_s\{Lz_k - \lambda_k H_z(x, z_k)\}\|_{E_s} \rightarrow 0, \\ \varepsilon_k := \int_M (H(x, z_k) - 1) dx \rightarrow 0 \end{cases} \quad (3.1)$$

as $k \rightarrow \infty$. Recalling $\int_M dx = 1$, from the assumption (H1) we derive

$$\int_M \langle H_z(x, z_k), z_k \rangle dx \geq 2(1 + \varepsilon_k) - c_0 \geq 2 - c_0 > 0 \quad \forall k \in \mathbb{N}. \quad (3.2)$$

Moreover, the definition of $d\mathcal{A}_H$ implies

$$\begin{aligned} & \langle d\mathcal{A}(z_k, \lambda_k), (z_k, \lambda_k) \rangle \\ &= (\mathcal{D}_s\{Lz_k - \lambda_k H_z(x, z_k)\}, z_k)_{E_s} - \lambda_k \int_M (H(x, z_k) - 1) dx \\ &= (Lz_k, z_k)_{L^2} - \lambda_k (H_z(x, z_k), z_k)_{L^2} - \lambda_k \int_M (H(x, z_k) - 1) dx \\ &= 2\mathcal{A}_H(z_k, \lambda_k) - \lambda_k (H_z(x, z_k), z_k)_{L^2} + \varepsilon_k \lambda_k \\ &= 2c - \lambda_k (H_z(x, z_k), z_k)_{L^2} + \varepsilon_k \lambda_k + o(1). \end{aligned} \quad (3.3)$$

Then it follows from (3.2) and (3.3) that for some constant $C > 0$ and all $k \in \mathbb{N}$,

$$\begin{aligned} |\lambda_k| &\leq \frac{1}{2 - c_0} |\lambda_k (H_z(x, z_k), z_k)_{L^2}| \\ &\leq \frac{1}{2 - c_0} \{|\varepsilon_k \lambda_k| + 2|\mathcal{A}_H(z_k, \lambda_k)| + |d\mathcal{A}_H(z_k, \lambda_k), (z_k, \lambda_k)|\} \\ &\leq \frac{1}{2 - c_0} \{|\varepsilon_k \lambda_k| + 2|\mathcal{A}_H(z_k, \lambda_k)| + \|d\mathcal{A}_H(z_k, \lambda_k)\|_{\mathcal{E}^*} \cdot \|(z_k, \lambda_k)\|_{\mathcal{E}}\} \\ &\leq C\{1 + |\varepsilon_k| |\lambda_k| + (\varepsilon_k + \varepsilon_k)(\|z_k\| + |\lambda_k|)\}. \end{aligned} \quad (3.4)$$

Next we estimate the E_s -norms of z_k . Since $\int_M H(x, z_k) dx = 1 + \varepsilon_k$ is bounded, by the Sobolev imbedding theorem and the assumption (H4), we get

$$\begin{aligned} \|\mathcal{D}_s H_z(x, z_k)\|_{E_s} &\leq \| |D|^{-2s} H_u(x, z_k) \|_{s,2} + \| |D|^{-2(1-s)} H_v(x, z_k) \|_{1-s,2} \\ &= \|H_u(x, z_k)\|_{-s,2} + \|H_v(x, z_k)\|_{-(1-s),2} \\ &\leq C(\|H_u(x, z_k)\|_{\frac{2n}{n+2s}} + \|H_v(x, z_k)\|_{\frac{2n}{n+2(1-s)}}) \leq C, \end{aligned} \quad (3.5)$$

where $C > 0$ denotes different constants. Note that the composition operator $\mathcal{D}_s L : E_s \rightarrow E_s$ is an isometry. Thus

$$\begin{aligned} \|z_k\|_{E_s}^2 &= \|\mathcal{D}_s Lz_k\|_{E_s}^2 \\ &= |(\mathcal{D}_s Lz_k, \mathcal{D}_s\{Lz_k - \lambda_k H_z(x, z_k)\})_{E_s} + (\mathcal{D}_s Lz_k, \lambda_k \mathcal{D}_s H_z(x, z_k))_{E_s}| \\ &\leq \varepsilon_k \|\mathcal{D}_s Lz_k\|_{E_s} + |\lambda_k| \|\mathcal{D}_s Lz_k\|_{E_s} \|\mathcal{D}_s H_z(x, z_k)\|_{E_s} \\ &\leq \varepsilon_k \|z_k\|_{E_s} + C|\lambda_k| \|z_k\|_{E_s}. \end{aligned} \quad (3.6)$$

Combining (3.4) with (3.6), we deduce that

$$|\lambda_k| + \|z_k\|_{E_s} \leq C\{1 + |\varepsilon_k| |\lambda_k| + \varepsilon_k(\|z_k\|_{E_s} + |\lambda_k|)\}, \quad (3.7)$$

which implies that both z_k and λ_k are bounded. Passing to a subsequence, we may assume that z_k converges weakly in E_s to $z = (u, v)$ and λ_k converges to $\lambda \in \mathbb{R}$. Let b_k be the first component

of $\nabla \mathcal{A}_H(z_k, \lambda_k)$, i.e., $b_k = \mathcal{D}_s L z_k - \lambda_k \mathcal{D}_s H_z(x, z_k)$, which converges to zero. Since the operator $\mathcal{D}_s L : E_s \rightarrow E_s$ is an isometry, and \mathcal{H}' is compact by Proposition 2.1, we conclude that

$$z_k = (\mathcal{D}_s L)^{-1} b_k - \lambda_k (\mathcal{D}_s L)^{-1} \mathcal{D}_s H_z(x, z_k) \quad (3.8)$$

converges in E_s and so $\|z_k - z\|_{E_s} \rightarrow 0$. This shows that \mathcal{A}_H satisfies the $(PS)_c$ condition. \square

Obverse that in the proof of boundedness of $\|(z_k, \lambda_k)\|_{\mathcal{E}}$ we only use the boundedness of $\mathcal{A}_H(z_k, \lambda_k)$ and the condition that $\|\nabla \mathcal{A}_H(z_k, \lambda_k)\|_{\mathcal{E}}$ is small enough. Consequently, we have

Corollary 3.2. *Suppose $|\mathcal{A}_H(z, \lambda)| < R$ for some constant $R > 0$. Then there exist $\epsilon > 0$ and $C = C(R)$ such that each $(z, \lambda) \in \mathcal{E}$ with $\|\nabla \mathcal{A}_H(z, \lambda)\|_{\mathcal{E}} \leq \epsilon$ satisfies $\|(z, \lambda)\|_{\mathcal{E}} < C$.*

3.2 Boundedness in \mathcal{E} for the autonomous flow

Since \mathcal{A}_H is of class C^2 under our assumptions, the local flow of $\nabla^K \mathcal{A}_H$ always exists.

Proposition 3.3. *Assume that $\mathbb{R} \ni t \mapsto (z(t), \lambda(t)) \in \mathcal{E}$ is a flow line of $-\nabla^K \mathcal{A}_H$ between critical points and that $\{\mathcal{A}_H(z(t), \lambda(t)) \mid t \in \mathbb{R}\} \subset [a, b]$ for some two real numbers $a < b$. Then there exists a constant $C_1 = C_1(R, a, b) > 0$ such that $\|(z(t), \lambda(t))\|_{\mathcal{E}} \leq C_1$ for all $t \in \mathbb{R}$.*

Proof. Let ϵ be as in Corollary 3.2 with $R = \max\{|a|, |b|\}$. For $s \in \mathbb{R}$, if $\|\nabla \mathcal{A}_H(z(s), \lambda(s))\|_{\mathcal{E}} < \epsilon$ we define $\tau(s) = 0$; otherwise, since $\|\nabla \mathcal{A}_H(z(s+t), \lambda(s+t))\|_{\mathcal{E}} \rightarrow 0$ as $t \rightarrow +\infty$ we have $\{t \geq 0 \mid \|\nabla \mathcal{A}_H(z(s+t), \lambda(s+t))\|_{\mathcal{E}} < \epsilon\} \neq \emptyset$ and hence

$$\tau(s) := \inf \left\{ t \geq 0 \mid \|\nabla \mathcal{A}_H(z(s+t), \lambda(s+t))\|_{\mathcal{E}} < \epsilon \right\} \quad (3.9)$$

is a nonnegative real number. Clearly, in the latter case it holds that

$$\|\nabla \mathcal{A}_H(z(s+t), \lambda(s+t))\|_{\mathcal{E}} \geq \epsilon \quad \forall t \in [s, s + \tau(s)].$$

Moreover, (2.18) implies that $((I + K)w, w)_{\mathcal{E}} \geq \frac{1}{2}\|w\|_{\mathcal{E}}^2$ for any $w \in \mathcal{E}$. Then

$$\begin{aligned} \|\nabla^K \mathcal{A}_H(z(t), \lambda(t))\|_{g^K}^2 &= ((I + K((z(t), \lambda(t))) \nabla \mathcal{A}_H(z(t), \lambda(t)), \nabla \mathcal{A}_H(z(t), \lambda(t)))_{\mathcal{E}} \\ &\geq \frac{1}{2} (\nabla \mathcal{A}_H(z(t), \lambda(t)), \nabla \mathcal{A}_H(z(t), \lambda(t)))_{\mathcal{E}}, \end{aligned}$$

and thus

$$\begin{aligned} b - a &\geq \mathcal{A}_H(z(-\infty), \lambda(-\infty)) - \mathcal{A}_H(z(+\infty), \lambda(+\infty)) \\ &= - \int_{-\infty}^{+\infty} \frac{d}{dt} \mathcal{A}_H(z(t), \lambda(t)) dt \\ &= \int_{-\infty}^{+\infty} \|\nabla^K \mathcal{A}_H(z(t), \lambda(t))\|_{g^K}^2 dt \\ &\geq \frac{1}{2} \int_{-\infty}^{+\infty} \|\nabla \mathcal{A}_H(z(t), \lambda(t))\|_{\mathcal{E}}^2 dt \\ &\geq \frac{1}{2} \int_s^{s+\tau(s)} \|\nabla \mathcal{A}_H(z(t), \lambda(t))\|_{\mathcal{E}}^2 dt \\ &\geq \frac{1}{2} \tau(s) \epsilon^2 \end{aligned} \quad (3.10)$$

if $\|\nabla \mathcal{A}_H(z(s), \lambda(s))\|_{\mathcal{E}} \geq \epsilon$. So for any $s \in \mathbb{R}$ we always have

$$\tau(s) \leq \frac{2(b-a)}{\epsilon^2}. \quad (3.11)$$

Moreover, $|A_H(z(s + \tau(s)), \lambda(s + \tau(s)))| \leq \max\{|a|, |b|\}$ and $\|\nabla A_H(z(s + \tau(s)), \lambda(s + \tau(s)))\|_{\mathcal{E}} \leq \epsilon$. It follows from Corollary 3.2 that

$$\|z(s + \tau(s))\|_{E_s} < C \quad \text{and} \quad |\lambda(s + \tau(s))| < C. \quad (3.12)$$

Since (2.18) implies that $((I + K)^{-1}w, w)_{\mathcal{E}} \geq \frac{2}{9}\|w\|_{\mathcal{E}}^2$ for any $w \in \mathcal{E}$, we get

$$\begin{aligned} \|\nabla^K \mathcal{A}_H(z(t), \lambda(t))\|_{\mathcal{E}}^2 &\leq \frac{9}{2}(\nabla^K \mathcal{A}_H(z(t), \lambda(t)), (I + K((z(t), \lambda(t)))^{-1} \nabla^K \mathcal{A}_H(z(t), \lambda(t)))_{\mathcal{E}} \\ &= \frac{9}{2}\|\nabla^K \mathcal{A}_H(z(t), \lambda(t))\|_{g^K}^2. \end{aligned}$$

This and $(z'(t), \lambda'(t)) = -\nabla^K \mathcal{A}_H(z(t), \lambda(t))$ lead to

$$\begin{aligned} \int_{-\infty}^{+\infty} \|z'(t)\|_{E_s}^2 dt + \int_{-\infty}^{+\infty} |\lambda'(t)|^2 dt &= \int_{-\infty}^{+\infty} \|\nabla^K \mathcal{A}_H(z(t), \lambda(t))\|_{\mathcal{E}}^2 dt \\ &\leq 5 \int_{-\infty}^{+\infty} \|\nabla^K \mathcal{A}_H(z(t), \lambda(t))\|_{g^K}^2 dt \\ &\leq 5(b-a). \end{aligned} \quad (3.13)$$

Using this, (3.11)-(3.12) and the Hölder inequality, we estimate

$$\begin{aligned} \|z(s)\| &\leq \|z(s + \tau(s))\| + \int_s^{s+\tau(s)} \|z'(t)\| dt \\ &\leq C + \sqrt{5(b-a)}\sqrt{\tau(s)} \\ &\leq C + \frac{5(b-a)}{\epsilon} := \frac{C_1}{\sqrt{2}} \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} |\lambda(s)| &\leq |\lambda(s + \tau(s))| + \int_s^{s+\tau(s)} |\lambda'(t)| dt \\ &\leq C + \sqrt{5(b-a)}\sqrt{\tau(s)} \leq \frac{C_1}{\sqrt{2}}. \end{aligned} \quad (3.15)$$

□

3.3 Boundedness for the non-autonomous flow

Suppose that $K \in C^0(\mathbb{R}, \mathbf{K})$ and $H \in C^2(\mathbb{R} \times \Sigma M \oplus \Sigma M)$ satisfy

- (i) $K_t(\cdot) := K(t, \cdot)$ is equal to K_0 for $t \leq 0$, and K_1 for $t \geq 1$;
- (ii) $H_t(\cdot) := H(t, \cdot)$ is equal to H_0 for $t \leq 0$, and H_1 for $t \geq 1$, and satisfies **(H1)** – **(H2)** and **(H4)** for all $t \in [0, 1]$;

(iii) there exists a constant A such that

$$\int_{-\infty}^{+\infty} \int_M \left| \frac{\partial H}{\partial t}(t, x, z(x)) \right| dx dt \leq A, \quad \forall z \in E_s. \quad (3.16)$$

For such a pair (H, K) we shall prove the boundedness on \mathcal{E} of solutions of the following nonautonomous system

$$\frac{dw(t)}{dt} + (I + K(t, w(t))) \nabla \mathcal{A}_H(t, x, w(t)) = 0. \quad (3.17)$$

Proposition 3.4. *Fix a pair (H, K) as above. Let $w(t) = (z(t), \lambda(t))$ be any solution of (3.17) with $\lim_{t \rightarrow \pm\infty} \mathcal{A}_{H_t}(z(t), \lambda(t)) \in [a, b]$ for some $a < b$, and let ϵ be as in Corollary 3.2. If $A < \epsilon/5$, then there exists a constant $C > 0$ only depending on a and b such that $|\lambda(t)| < C$ and $\|z(t)\|_{E_s} < C$ for any $t \in \mathbb{R}$.*

Proof. For a fixed t , we denote the gradient of \mathcal{A}_{H_t} with respect to g^{K_t} by $\nabla^{K_t} \mathcal{A}_{H_t}$. Since

$$\frac{d}{dt} \mathcal{A}_{H_t}(w(t)) = -\|\nabla^{K_t} \mathcal{A}_{H_t}(w(t))\|_{g^K} - \lambda(t) \int_M \frac{\partial H}{\partial t}(t, x, z(t)) dx, \quad (3.18)$$

we have

$$\begin{aligned} & \int_{-\infty}^{+\infty} \|\nabla^{K_t} \mathcal{A}_{H_t}(w(t))\|_{g^K}^2 dt \\ & \leq \lim_{t \rightarrow -\infty} \mathcal{A}_{H_t}(w(t)) - \lim_{t \rightarrow +\infty} \mathcal{A}_{H_t}(w(t)) + \int_M \int_{-\infty}^{+\infty} |\lambda(t)| \left| \frac{\partial H}{\partial t}(t, x, z) \right| dt dx \\ & \leq b - a + A \|\lambda\|_{\infty}. \end{aligned} \quad (3.19) \quad (3.20)$$

Defining $\tau(s)$ for $s \in \mathbb{R}$ as in the proof of Proposition 3.3, we have

$$\begin{aligned} \epsilon^2 \tau(s) & \leq \int_s^{s+\tau(s)} \|\nabla \mathcal{A}_{H_t}(w(t))\|_{\mathcal{E}}^2 dt \\ & \leq 2 \int_s^{s+\tau(s)} \|\nabla^{K_t} \mathcal{A}_{H_t}(w(t))\|_{g^K}^2 dt \\ & \leq 2(b - a + A \|\lambda\|_{\infty}), \end{aligned} \quad (3.21)$$

and hence

$$\tau(s) \leq \frac{2(b - a + A \|\lambda\|_{\infty})}{\epsilon^2}. \quad (3.22)$$

Then as in (3.13) we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \|z'(t)\|_{E_s}^2 dt + \int_{-\infty}^{+\infty} |\lambda'(t)|^2 dt & = \int_{-\infty}^{+\infty} \|\nabla^K \mathcal{A}_{H_t}(z(t), \lambda(t))\|_{\mathcal{E}}^2 dt \\ & \leq 5 \int_{-\infty}^{+\infty} \|\nabla^K \mathcal{A}_{H_t}(z(t), \lambda(t))\|_{g^K}^2 dt \\ & \leq 5(b - a + A \|\lambda\|_{\infty}). \end{aligned}$$

This and (3.22) lead to

$$\begin{aligned}
|\lambda(s)| &= \left| \lambda(s + \tau(s)) + \int_s^{s+\tau(s)} \lambda'(t) dt \right| \\
&\leq |\lambda(s + \tau(s))| + \int_s^{s+\tau(s)} \|\nabla \mathcal{A}_{H_t}^K(w(t))\|_{\mathcal{E}} dt \\
&\leq C + \sqrt{2\tau(s)} \sqrt{5(b-a + A\|\lambda\|_{\infty})} \\
&\leq C + \frac{5(b-a + A\|\lambda\|_{\infty})}{\epsilon} \quad \forall s \in \mathbb{R}.
\end{aligned}$$

It follows from $A < \epsilon/5$ that

$$\|\lambda\|_{\infty} \leq \frac{C\epsilon + 5(b-a)}{\epsilon - 5A}.$$

Similarly, we have the uniform bound for $\|z(t)\|$. In fact, the following estimate holds

$$\|z(s)\| \leq \|z(s + \tau(s))\| + \frac{5(b-a + A\|\lambda\|_{\infty})}{\epsilon}.$$

From Corollary 3.2, we obtain $\|z(s + \tau(s))\| \leq C$ and thus $\|z(t)\|$ is uniformly bounded. \square

3.4 Boundness in Hölder spaces

In Proposition 3.4 we get an uniform bound for all nonautonomous flows under certain conditions. Following the ideas in [7] we shall show that $z(t)$ is also uniformly bounded in Hölder spaces $C^{i,\alpha}(M, \Sigma M)$, $i = 0, 1$ (for the explicit definition see [6, Chapter 3]) through the bootstrap argument.

Proposition 3.5. *Assume H is as in Proposition 3.4. Let $w(t) = (z(t), \lambda(t))$ be a solution of (3.17). Then $z(t)$ is bounded in $C^{\alpha}(M, \Sigma M) \oplus C^{\beta}(M, \Sigma M)$ with $0 < \alpha < \min\{1, 2s\}$ and $0 < \beta < \min\{1, 2(1-s)\}$. In particular, for some constant $C > 0$ we have*

$$\sup_{-\infty < t < \infty} \|z(t)\|_{C^{\alpha} \oplus C^{\beta}} < C \sup_{-\infty < t < \infty} \|z(t)\|_{E_s}.$$

For $0 < \alpha < 1$ (resp. $1 < \alpha < 2$), let $C^{\alpha}(M, \Sigma M)$ represent Hölder spaces $C^{0,\alpha}(M, \Sigma M)$ (resp. $C^{1,\alpha-1}(M, \Sigma M)$).

In order to prove the above proposition we give two lemmas.

Lemma 3.6. *Let $0 < r < \infty$ and $1 < \alpha, \beta < \infty$. Assume $u \in L^{\alpha}(M, \Sigma M)$.*

If $\alpha < \frac{n}{r}$ then $|D|^{-r}u \in L^{\beta}(M, \Sigma M)$ with $\frac{1}{\beta} = \frac{1}{\alpha} - \frac{r}{n}$.

If $\alpha > \frac{n}{r}$ then $|D|^{-r}u \in C^{r-\frac{n}{\alpha}}(M, \Sigma M)$.

Since $\mathcal{D}_s L : E_s \rightarrow E_s$ is an self-adjoint Fredholm isometry operator and

$$(\mathcal{D}_s L) \circ (\mathcal{D}_s L) = \begin{pmatrix} |D|^{-2s} D |D|^{-2(1-s)} D & 0 \\ 0 & |D|^{-2(1-s)} D |D|^{-2s} D \end{pmatrix} = \text{Id}_{E_s},$$

we can split E_s into

$$E_s = E_+ \oplus E_- \tag{3.23}$$

with $(\mathcal{D}_s L)|_{E_\pm} = \pm \text{Id}_{E_\pm}$. Denote by P_\pm projections onto the eigenspace E_\pm respectively. Then the evolution operator $\frac{d}{dt} + \mathcal{D}_s L$ has a fundamental solution operator

$$G(t) = e^{-t} \chi_{\mathbb{R}_+}(t) P_- - e^t \chi_{\mathbb{R}_-}(t) P_+,$$

which gives us a formulation for any solution of (3.17)

$$z(t) = \int_{-\infty}^{+\infty} G(t-s) \left\{ \lambda(s) \mathcal{D}_s H_z(s, x, z(s)) + \text{Pr}_1(K(s, z(s)) \nabla \mathcal{A}_H)(w(s)) \right\} ds,$$

where $\text{Pr}_1(K(s, z(s)) \nabla \mathcal{A}_H(w(s)))$ is uniformly bounded in $C^2(M, \Sigma M \oplus \Sigma M)$ by (2.24) and Proposition 3.4. To show that the operator $G(t) \mathcal{D}_s$ has a regularizing effect, note that the above projections P_\pm onto E_\pm can be written as

$$P_\pm = \frac{1}{2} \begin{pmatrix} I_+ & \pm |D|^{-2s} D \\ \pm |D|^{-2(1-s)} D & I_- \end{pmatrix},$$

where I_\pm represents the identity operator on E_\pm respectively. It is not hard to estimate

$$|G(t) \mathcal{D}_s H_z(t, x, z(t))| \leq \frac{1}{2} e^{-|t|} \begin{pmatrix} |D|^{-2s} |H_u| + |D|^{-1} |H_v| \\ |D|^{-1} |H_u| + |D|^{-2(1-s)} |H_v| \end{pmatrix}. \quad (3.24)$$

Using this estimate, the assumption (H2) and Lemma 3.6, we obtain the bootstrap lemma

Lemma 3.7. *Let $(z(t), \lambda(t))$ be a solution of (3.17) with $[z]_{\alpha, \beta} < \infty$, where*

$$z(t) = (u(t, \cdot), v(t, \cdot)) \quad \text{and} \quad [z]_{\alpha, \beta} := \sup_{-\infty < t < +\infty} \{ \|u(t, \cdot)\|_{L^\alpha} + \|v(t, \cdot)\|_{L^\beta} \}.$$

Suppose that two real numbers α^ and β^* are defined by*

$$\begin{cases} \frac{1}{\alpha^*} = \max \left\{ \frac{p}{\alpha} - \frac{2s}{n}, \frac{p(q+1)}{\beta(p+1)} - \frac{2s}{n}, \frac{q(p+1)}{\alpha(q+1)} - \frac{1}{n}, \frac{q}{\beta} - \frac{1}{n} \right\} \\ \frac{1}{\beta^*} = \max \left\{ \frac{p}{\alpha} - \frac{1}{n}, \frac{p(q+1)}{\beta(p+1)} - \frac{1}{n}, \frac{q(p+1)}{\alpha(q+1)} - \frac{2(1-s)}{n}, \frac{q}{\beta} - \frac{2(1-s)}{n} \right\}. \end{cases} \quad (3.25)$$

If $\alpha^, \beta^* > 0$, then $[z]_{\alpha^*, \beta^*} < \infty$.*

If $\alpha^ < 0$, then $u(t, \cdot)$ is uniformly bounded in $C^{-\frac{1}{\alpha^*}}(M, \Sigma M)$.*

If $\beta^ < 0$, then $v(t, \cdot)$ is uniformly bounded in $C^{-\frac{1}{\beta^*}}(M, \Sigma M)$.*

Proof of Proposition 3.5. Since $z(t)$ is uniformly bounded in E_s , we have $[z]_{\alpha_1, \beta_1} < \infty$ with $\alpha_1 = \frac{2n}{n-2s}$ and $\beta_1 = \frac{2n}{n-2(1-s)}$ because of the Sobolev embedding theorems. By Lemma 3.7 the condition $[z]_{\alpha, \beta} < \infty$ implies $[z]_{\alpha^*, \beta^*} < \infty$. It follows from the subcritical condition (1.4) that $\alpha^* > \alpha$ and $\beta^* > \beta$ whenever $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$. After k times of iterations one gets $[z]_{\alpha_k, \beta_k} < \infty$ with $\alpha_{k+1} < 0$ and $\beta_{k+1} < 0$. Using the bootstrap lemma again we arrive at the conclusion of Proposition 3.5. \square

4 Relative index and Moduli space of trajectories

4.1 Morse functions and the relative index

According to the usual definition of the Morse index of a critical point, the Rabinowitz-Floer functional \mathcal{A}_H has infinite index and co-index at each critical point. In order to give a natural grading of the Rabinowitz-Floer homology groups we will adopt the way of Abbondandolo [3].

If (z, λ) is a critical of \mathcal{A}_H , then the Hessian of \mathcal{A}_H at (z, λ) can be written as

$$\text{Hess}\mathcal{A}_H(z, \lambda) = \begin{pmatrix} \mathcal{D}_s L - \lambda \mathcal{D}_s H_{zz}(x, z) & -\mathcal{D}_s H_z(x, z) \\ -(\mathcal{D}_s H_z(x, z))^* & 0 \end{pmatrix}.$$

Definition 4.1. A critical point (z, λ) of \mathcal{A}_H is called *nondegenerate* if the Hessian operator $\text{Hess}\mathcal{A}_H(z, \lambda) : \mathcal{E} \rightarrow \mathcal{E}$ is a bounded linear bijective operator. The functional \mathcal{A}_H is said to be *Morse* if all critical points of it are nondegenerate.

Remark 4.2. In our case, we always view \mathcal{E} as a real Hilbert space with the inner product $(\cdot, \cdot)_{\mathcal{E}}$ defined by (2.11). In the following, for a subspace V of \mathcal{E} , $\dim(V)$ stands for the real dimension of V .

Definition 4.3. Let U and V be two closed subspaces in a Hilbert space H . Denote P_U and P_V the orthogonal projections onto U and V respectively. If $P_U - P_V$ is compact on H , U is said to be *commensurable* to V , and define the relative dimension of U and V as

$$\dim(U, V) = \dim(U \cap V^\perp) - \dim(U^\perp \cap V).$$

One can check that it is well-defined and finite. Moreover, if U , V and W are three each other commensurable closed subspaces in H , then

$$\dim(U, V) = \dim(U, W) + \dim(W, V) \quad (4.1)$$

(cf. [3, Section 2]).

Lemma 4.4. Suppose A and B are self-adjoint Fredholm operators on a Hilbert space H with $A - B$ compact. Then the negative (resp. positive) eigenspaces of B is commensurable with the negative (resp. positive) eigenspaces of A .

Definition 4.5. Let E_- be as in (3.23). Denote $V^-(z, \lambda)$ the maximal negative definite subspace in \mathcal{E} of the Hessian of \mathcal{A}_H at a critical point (z, λ) . We define the relative index as

$$i_{\text{rel}}(z, \lambda) = \dim(V^-(z, \lambda), E_- \times \{0\}). \quad (4.2)$$

Lemma 4.6. Under the assumptions (H1)–(H4), the relative index is well defined for all critical points of \mathcal{A}_H .

Proof. Let (z, λ) be a critical point of \mathcal{A}_H . Notice that $\mathcal{D}_s L \oplus Id_{\mathbb{R}}$ and $\text{Hess} \mathcal{A}_H(z, \lambda)$ are both self-adjoint Fredholm operators on \mathcal{E} respectively, where $Id_{\mathbb{R}}$ is the identity operator on \mathbb{R} . Moreover, $E_- \times \{0\}$ and $V^-(z, \lambda)$ are the negative eigenspaces of $\mathcal{D}_s L \oplus Id_{\mathbb{R}}$ and $\text{Hess} \mathcal{A}_H(z, \lambda)$, respectively. It follows from Proposition 2.1 that

$$\text{Hess} \mathcal{A}_H(z, \lambda) - \mathcal{D}_s L \oplus Id_{\mathbb{R}} = \begin{pmatrix} -\lambda \mathcal{D}_s H_{zz}(x, z) & -\mathcal{D}_s H_z(x, z) \\ -(\mathcal{D}_s H_z(x, z))^* & -1 \end{pmatrix} \quad (4.3)$$

is compact on \mathcal{E} . Hence Lemma 4.4 implies that the relative index $i_{\text{rel}}(z, \lambda)$ is well defined. \square

4.2 Moduli space of trajectories and grading

Let $w_1 = (z_1, \lambda_1)$ and $w_2 = (z_2, \lambda_2)$ be two nondegenerate critical points of \mathcal{A}_H , and let a and b be two real numbers such that $a \leq \mathcal{A}_H(w_i) \leq b$, $i = 1, 2$. Define the space of connecting orbits from w_1 to w_2 as

$$\mathcal{M}_{H,K}^{a,b}(w_1, w_2) = \left\{ w \in C^1(\mathbb{R}, \mathcal{E}) \left| \begin{array}{l} w(t) \text{ is a solution of (2.23) with} \\ w_1 = w(-\infty) =: \lim_{t \rightarrow -\infty} w(t) \\ \text{and } w_2 = w(+\infty) =: \lim_{t \rightarrow +\infty} w(t) \end{array} \right. \right\}, \quad (4.4)$$

and the moduli space of trajectories $\widehat{\mathcal{M}}_{H,K}^{a,b}(w_1, w_2)$ to be the quotient of $\mathcal{M}_{H,K}^{a,b}(w_1, w_2)$ with respect to the free action of \mathbb{R} given by the flow of $-\nabla^K \mathcal{A}_H$. Denote by $\hat{w} \in \widehat{\mathcal{M}}_{H,K}^{a,b}(w_1, w_2)$ the unparametrized trajectory corresponding to $w \in \mathcal{M}_{H,K}^{a,b}(w_1, w_2)$. Let

$$Q_1 = \overline{w} + W^{1,2}(\mathbb{R}, \mathcal{E}) \quad \text{and} \quad Q_0 = L^2(\mathbb{R}, \mathcal{E}),$$

where $\overline{w} : \mathbb{R} \rightarrow \mathcal{E}$ is a fixed smooth map which satisfies $\overline{w}(t) \equiv w_1$ for $t \leq -1$ and $\overline{w}(t) \equiv w_2$ for $t \geq 1$. The space of parametrized trajectories $\mathcal{M}_{H,K}^{a,b}(w_1, w_2)$ can be considered as the zero set of the map $\mathcal{F}_{H,K} : Q_1 \rightarrow Q_0$ given by

$$\mathcal{F}_{H,K}(w) = \frac{dw}{dt} + \nabla^K \mathcal{A}_H(w). \quad (4.5)$$

Suppose that \mathcal{A}_H is a Morse function. To determine the dimension of the moduli space of trajectories, we need that (H, K) satisfies the so-called *Morse-Smale condition*, that is to say, for every pair of critical points w_1, w_2 of \mathcal{A}_H the unstable manifold $W^u(w_1; -\nabla^K \mathcal{A}_H)$ is transverse to the stable manifold $W^s(w_2; -\nabla^K \mathcal{A}_H)$, i.e.,

$$T_w W^u(w_1) + T_w W^s(w_2) = \mathcal{E}, \quad \forall w \in W^u(w_1) \cap W^s(w_2).$$

The pair (H, K) with H satisfying (H1) – (H4) and $K \in \mathbf{K}$ is called *regular* if \mathcal{A}_H is Morse and (H, K) satisfies the Morse-Smale condition. Denote by Ω_{reg} the set consisting of all such regular pairs.

Lemma 4.7 ([7, Proposition 12]). *Let E be a Banach space. Let $B : \mathbb{R} \rightarrow \mathcal{L}(E, E)$ be a continuous map such that each $B(t)$ is compact operator and that $\lim_{t \rightarrow \pm\infty} B(t) = 0$. Then the operator $M_B : W^{1,2}(\mathbb{R}, E) \rightarrow L^2(\mathbb{R}, E)$ given by*

$$M_B a(t) = B(t)a(t) \quad \text{for } a \in W^{1,2}(\mathbb{R}, E)$$

is compact.

Lemma 4.8 ([3, Theorem 3.4]). *Let E be a Banach space. Let $A : \mathbb{R} \rightarrow L(E, E)$ be continuous map such that*

$$A(-\infty) := \lim_{t \rightarrow -\infty} A(t) \quad \text{and} \quad A(+\infty) := \lim_{t \rightarrow +\infty} A(t)$$

exist and are self-adjoint and invertible. Then the operator

$$L_A : W^{1,2}(\mathbb{R}, E) \rightarrow L^2(\mathbb{R}, E)$$

given by $L_A w(t) = w'(t) + A(t)w(t)$, is a Fredholm operator of index

$$\text{ind} L_A = \dim(V^-(A(-\infty)), V^-(A(+\infty))),$$

where $V^-(L)$ represents the negative eigenspace of L on E .

Proposition 4.9. *Let (H, K) be a regular pair. Suppose that the relative indexes of two critical points w_1 and w_2 of \mathcal{A}_H satisfy $i_{\text{rel}}(w_1) - i_{\text{rel}}(w_2) > 0$. Then $\widehat{\mathcal{M}}_{H,K}^{a,b}(w_1, w_2)$ is a manifold of dimension $i_{\text{rel}}(w_1) - i_{\text{rel}}(w_2) - 1$ if $\mathcal{M}_{H,K}^{a,b}(w_1, w_2) \neq \emptyset$.*

Proof. Let $K_\theta(w) = I + \theta K(w)$ for any $w \in \mathcal{E}$, where $\theta \in [0, 1]$. Then the linearized operator of $\mathcal{F}_{H,\theta K}$ at $w = (z, \lambda)$ is given by

$$D\mathcal{F}_{H,\theta K}(w) = \frac{d}{dt} + K_\theta(w(t))\text{Hess}\mathcal{A}_H(w(t)) + \{dK_\theta(w(t))[\cdot]\}\nabla\mathcal{A}_H(w(t)). \quad (4.6)$$

We shall use the implicit function theorem to prove our result. To this end we need to show that $D\mathcal{F}_{H,K}(w)$ is Fredholm and onto. Notice that $\text{Hess}\mathcal{A}_H(w)$ has the following decomposition

$$\text{Hess}\mathcal{A}_H(z, \lambda) = \begin{pmatrix} \mathcal{D}_s L & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\lambda \mathcal{D}_s H_{zz}(x, z) & -\mathcal{D}_s H_z(x, z) \\ -(\mathcal{D}_s H_z(x, z))^* & -1 \end{pmatrix}.$$

It follows from Proposition 2.1 that $\text{Hess}\mathcal{A}_H(w)$ is a compact perturbation of $\begin{pmatrix} \mathcal{D}_s L & 0 \\ 0 & 1 \end{pmatrix}$ with spectrum $\{-1, 1\}$. For a fixed $t \in \mathbb{R}$, by definition the linear maps

$$T(t) : \mathcal{E} \rightarrow C^2(M, \Sigma M \oplus \Sigma M) \times \mathbb{R}, \quad \xi \mapsto \{dK_\theta(w(t))[\xi]\}\nabla\mathcal{A}_H(w(t))$$

is bounded, and hence compact as an operator from \mathcal{E} to \mathcal{E} . Since $\lim_{t \rightarrow \pm\infty} \nabla\mathcal{A}_H(w(t)) = 0$, Lemma 4.7 implies that the third term in (4.6) is compact. Therefore $D\mathcal{F}_{H,\theta K}(w)$ is a family of

Fredholm operator from $W^{1,2}(\mathbb{R}, \mathcal{E})$ to $L^2(\mathbb{R}, \mathcal{E})$. By the homotopy invariance of Fredholm index (cf. [31]) and Lemma 4.8, we obtain

$$\begin{aligned}
 \text{ind}(D\mathcal{F}_{H,K}(w)) &= \text{ind}(D\mathcal{F}_{H,0}(w)) \\
 &= \dim(V^-(w_1), E_- \times \{0\}) + \dim(E_- \times \{0\}, V^-(w_2)) \\
 &= \dim(V^-(w_1), E_- \times \{0\}) - \dim(V^-(w_2), E_- \times \{0\}) \\
 &= i_{\text{rel}}(w_1) - i_{\text{rel}}(w_2).
 \end{aligned}$$

On the other hand, the Morse-Smale condition guarantees that $D\mathcal{F}_{H,K}(w)$ is onto (see [2] for a detailed proof). So $\mathcal{M}_{H,K}^{a,b}(w_1, w_2)$ is a manifold of dimension $i_{\text{rel}}(w_1) - i_{\text{rel}}(w_2)$ and thus the desired result is obtained by modeling the free \mathbb{R} -action. \square

4.3 Broken trajectories and gluing

By Proposition 3.5, for any $w = (z, \lambda) \in \mathcal{M}_{H,K}(w_1, w_2)$ with $\sup_t |\mathcal{A}_H(w(t))| < \infty$, $\lambda(t)$ is bounded in \mathbb{R} and $z(t)$ belongs to a compact subset of

$$X_{\alpha,\beta} := C^\alpha(M, \Sigma M) \oplus C^\beta(M, \Sigma M)$$

for some two constants $0 < \alpha < \min\{1, 2s\}$, $0 < \beta < \min\{1, 2(1-s)\}$. As a result, the moduli spaces are modeled on the affine space

$$\mathcal{Q}^1 := C^1(\mathbb{R}, X), \quad \text{where } X = X_{\alpha,\beta} \times \mathbb{R}.$$

In the following, we will equip the $C_{loc}^0(\mathbb{R}, X)$ -topology on the moduli space $\widehat{\mathcal{M}}_{H,K}(w_1, w_2)$, i.e., the uniform X -norm convergence on bounded intervals of \mathbb{R} .

Definition 4.10. A broken trajectory from $x \in \text{Crit}(\mathcal{A}_H)$ to $y \in \text{Crit}(\mathcal{A}_H)$ consists of critical points $x = w_0, w_1, \dots, w_k = y$, $k \geq 2$ and a tuple $(\hat{u}_1, \dots, \hat{u}_k)$ of unparametrized flow lines of $-\nabla^K \mathcal{A}_H$ such that $u_{i-1}(-\infty) = w_{i-1}$ and $u_{i-1}(+\infty) = w_i$, $1 \leq i \leq k$.

The evaluation map $EV : \mathcal{M}_{H,K}(w_1, w_2) \rightarrow X$ defined by

$$EV(w) = w(0)$$

is a continuous injective map. It has also a precompact image set. In fact, let $(w^i)_{i=1}^\infty$ be a sequence in $\mathcal{M}_{H,K}(w_1, w_2)$. Write $EV(w^i) = w_0^i$ and denote the corresponding orbits by $c^i := w^i(\mathbb{R})$. As a result of the compactness before, the union set $\bigcup_i c^i \cup \{w_1, w_2\}$ constitutes a compact set in X . Under the assumption that the critical points of \mathcal{A}_H are isolated, a standard argument ([7, Section 5]) shows that the sequence $\bar{c}^i = c^i \cup \{w_1, w_2\}$ has a subsequence converging to some compact set c^* in X which is either a broken trajectory or the union of an unparametrized flow line and the set $\{w_1, w_2\}$.

The above argument tells us that moduli spaces of trajectories are generally not compact. In the following we will use a gluing construction similar to that in Morse homology [40, Chapter 2] to show that the closure of $EV(\mathcal{M}_{H,K}(w_1, w_2))$ is compact in X .

Suppose that $w_{12} \in \mathcal{M}_{H,K}(w_1, w_2)$ and $w_{23} \in \mathcal{M}_{H,K}(w_2, w_3)$ with $i_{rel}(w_1) > i_{rel}(w_2) > i_{rel}(w_3)$. Since both $D\mathcal{F}_{H,K}(w_{12})$ and $D\mathcal{F}_{H,K}(w_{23})$ are Fredholm and onto provided that (H, K) satisfies the Morse-Smale condition, they have bounded right inverses Ψ_{12} and Ψ_{23} from $L^2(\mathbb{R}, \mathcal{E})$ to $W^{1,2}(\mathbb{R}, \mathcal{E})$, respectively. Let ζ be a nonnegative smooth function such that

$$\zeta(t) \equiv 0 \quad \text{for } t \leq -1, \quad \zeta(t) \equiv 1 \quad \text{for } t \geq 1, \quad \text{and } \zeta'(t) \geq 0 \quad \forall t.$$

Let $\zeta_T(t) = \zeta(\frac{t}{T})$ for large $T > 0$. Now we glue w_{12} and w_{23} as follow:

$$w_{13,T}(t) = (1 - \zeta_T(t))w_{12}(t + 2T) + \zeta_T(t)w_{23}(t - 2T), \quad t \in \mathbb{R}.$$

Accordingly, we define the gluing operator

$$\Phi_T = \rho_T^+ \tau_{2T} \Psi_{12} \tau_{-2T} \rho_T^+ + \rho_T^- \tau_{2T} \Psi_{23} \tau_{-2T} \rho_T^-,$$

where τ_s is a translation operator satisfying $\tau_h g(t) = g(t+h)$ and ρ^\pm is a pair of smooth functions satisfying

$$\begin{aligned} \rho_T^\pm &= \rho^\pm\left(\frac{t}{T}\right), \quad (\rho_1^+)^2 + (\rho_1^-)^2 = 1, \\ \rho_1^+(t) &= 0 \quad \text{for } t \leq -1, \quad \rho_1^-(t) = \rho_1^+(-t). \end{aligned}$$

A direct computation yields that $D\mathcal{F}_{H,K}(w_{13,T}) \circ \Phi_T$ converges to the identity operator on $L^2(\mathbb{R}, \mathcal{E})$ as $T \rightarrow \infty$. Then it follows from the implicit function theorem that the equation

$$\mathcal{F}_{H,K}(w) = 0$$

has solutions of the form $w = w_{13,T} + \Phi_T \eta$ with $\eta \in L^2(\mathbb{R}, \mathcal{E})$ whenever T is large enough. Moreover, such a solution w with small $\eta \in L^2(\mathbb{R}, \mathcal{E})$ is unique, denoted by $w_{12} \#_T w_{23}$. Hence one can approximate the broken trajectory $(\hat{w}_{12}, \hat{w}_{23})$ by such C^1 -glued orbits. Since these glued orbits represent approximate solutions of the negative gradient flow in a unique way for large gluing parameter T , from the fact that $EV(\mathcal{M}_{H,K}(w_1, w_2))$ has compact closure we deduce that the moduli space $\widehat{\mathcal{M}}_{H,K}(w_1, w_2)$ is finite whenever $i_{rel}(w_1) = i_{rel}(w_2) + 1$.

5 The Rabinowitz Floer complex of \mathcal{A}_H

In former two subsections we shall give the construction of the Rabinowitz Floer homology of \mathcal{A}_H in Morse situation and Morse-Bott one. Then we prove continuation invariance of the homology in the Morse case in Section 5.3.

5.1 The Morse situation

Given a pair $(H, K) \in \Omega_{reg}$, $k \in \mathbb{N}$ and real numbers $a < b$, let $\text{Crit}(\mathcal{A}_H)$ be the set of critical points of the functional \mathcal{A}_H , $\text{Crit}_k(\mathcal{A}_H) = \{x \in \text{Crit}(\mathcal{A}_H) \mid i_{\text{rel}}(x) = k\}$ and

$$\begin{aligned}\text{Crit}^{[a,b]}(\mathcal{A}_H) &:= \{x \in \text{Crit}(\mathcal{A}_H) \mid a \leq \mathcal{A}_H(x) \leq b\}, \\ \text{Crit}_k^{[a,b]}(\mathcal{A}_H) &:= \{x \in \text{Crit}_k(\mathcal{A}_H) \mid a \leq \mathcal{A}_H(x) \leq b\}.\end{aligned}$$

Denote by $\text{CF}_k(H)$ the chain complex as the vector space over \mathbb{Z}_2 generated by $\text{Crit}_k(\mathcal{A}_H)$. It needs not to be finitely generated. But its \mathbb{Z}_2 -subspace $\text{CF}_k^{[a,b]}(H)$ generated by $\text{Crit}_k^{[a,b]}(\mathcal{A}_H)$ has only finite elements since \mathcal{A}_H is Morse and satisfies the $(PS)_c$ condition. As we said before, if $x, y \in \text{Crit}(\mathcal{A}_H)$ satisfy $i_{\text{rel}}(x) = i_{\text{rel}}(y) + 1$, then the integer $\#\widehat{\mathcal{M}}_{H,K}(x, y)$ is finite. Define

$$n_2(x, y) := \#\widehat{\mathcal{M}}_{H,K}(x, y) \bmod [2],$$

and the boundary operator $\partial_k : \text{CF}_k^{[a,b]}(H) \rightarrow \text{CF}_{k-1}^{[a,b]}(H)$ by

$$\partial_k \left(\sum_{j=1}^r m_j x_j \right) = \sum_{j=1}^r \sum_{y \in \text{CF}_{k-1}^{[a,b]}(H)} m_j n_2(x_j, y) y \quad (5.1)$$

for $\sum_{j=1}^r m_j x_j \in \text{CF}_k^{[a,b]}(H)$. Due to the gluing argument of Section 4.3, we have $\partial_k \partial_{k-1} = 0$. So $(\text{CF}_*^{[a,b]}(H), \partial_*)$ is a chain complex which is called the *Rabinowitz Floer complex* of \mathcal{A}_H . The corresponding homology is called the *Rabinowitz Floer homology* of \mathcal{A}_H defined by

$$RHF_k^{[a,b]}(H, K) = \frac{\ker(\partial_k)}{\text{im}(\partial_{k+1})}.$$

5.2 The Morse-Bott situation

In this subsection, we construct the Rabinowitz Floer homology when the functional \mathcal{A}_H is Morse-Bott.

Definition 5.1. Let B be a Hilbert space. A C^k functional f ($k \geq 2$) on a Hilbert space B is called *Morse-Bott* if its critical set

$$\text{Crit}(f) = \{x \in B \mid df(x) = 0\}$$

is a C^{k-1} submanifold (possible with components of different dimensions) and it holds that

$$T_x \text{Crit}(f) = \ker(\text{Hess}(f)(x)) \quad \forall x \in \text{Crit}(f).$$

Suppose that H is invariant under the action of S^1 on $\Sigma M \times \Sigma M$, that is,

$$H(x, e^{i\theta} z) = H(x, z), \quad \forall \theta \in \mathbb{R}, \quad \forall x \in M, \quad \forall z = (u, v) \in \Sigma_x M \times \Sigma_x M.$$

By extending the S^1 action on \mathbb{R} trivially, we find that \mathcal{A}_H is also invariant under this action, i.e.,

$$\mathcal{A}_H(e^{i\theta} z, e^{i\theta} \lambda) = \mathcal{A}_H(z, \lambda), \quad \forall z \in E_s, \quad \forall \lambda \in \mathbb{R}.$$

In this case, the functional \mathcal{A}_H is never Morse. One way to overcome it is to choose an additional small perturbation to reduce to the Morse situation as before. However, even for the nonlinearity $H(x, u, v) = \frac{1}{2}(u^2 + v^2)$, giving an explicit computation of the Rabinowitz Floer homology of it needs an elaborate perturbation and a good computation for the index. It seems to be difficult for the authors to carry out them.

Instead we may assume that \mathcal{A}_H is Morse-Bott, and then choose a Morse function satisfying the Morse-Smale condition on the critical manifold. The chain complex is generated by the critical points of this Morse function, while the boundary operator is defined by counting flow lines with cascades. For the definition of gradient flow lines with cascades we refer to Frauenfelder [20] or [13, 17].

5.2.1. Flow lines with cascades. Assume that $\text{Crit}(\mathcal{A}_H)$ is a C^2 -submanifold of \mathcal{E} . We choose a Riemannian metric g on $\text{Crit}(\mathcal{A}_H)$ and a Morse function $h : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{R}$, such that (h, g) satisfy the Morse-Smale condition, that is, for every pair of critical points x, y of h , the unstable manifold $W^u(x; -\nabla h)$ is transverse to the stable manifold $W^s(y; -\nabla h)$. Let $\text{ind}(x)$ denote the Morse index of h at x , i.e., $\text{ind}(x) = \dim W^u(x; -\nabla h)$. We now assign an index ν to x by

$$\nu(x) := i_{\text{rel}}(x) + \text{ind}(x).$$

Given two real numbers $a < b$, we further assume

the critical manifold of \mathcal{A}_H containing in $\mathcal{A}_H^{-1}[a, b]$ is the union of a finite number of disjoint, compact, non-degenerate critical manifolds.

Let $\Psi_h(t) \in \text{Diff}(\text{Crit}^{[a,b]}(\mathcal{A}_H))$ be the smooth family of diffeomorphisms defined by

$$\Psi_h(t)(p) = \varphi_p(t), \quad \forall p \in \text{Crit}^{[a,b]}(\mathcal{A}_H),$$

where φ_p is the flow line of $-\nabla h$ with $\varphi_p(0) = p$. Let σ_0 and σ_1 be two components of $\text{Crit}^{[a,b]}(\mathcal{A}_H)$, and $x_i \in \sigma_i$, $i = 0, 1$, and an integer $m \geq 1$. A *flow line from x_0 to x_1 with m cascades* consists of $m - 1$ components c_1, \dots, c_{m-1} of $\text{Crit}(\mathcal{A}_H)$ and a m tuple (w_1, \dots, w_m) of solutions of the Rabinowitz-Floer equation (2.23) such that

$$\begin{aligned} w_1(-\infty) &\in W^u(x_0; -\nabla h), \quad w_m(+\infty) \in W^s(x_1; -\nabla h), \\ w_{j+1}(-\infty) &= \Phi_h(t_j)(w_j(+\infty)) \quad \text{for some } t_j \in \mathbb{R}_+, \quad j = 1, \dots, m-1, \end{aligned}$$

where $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$. Denote by

$$\mathcal{M}_m^{a,b}(x_0, x_1; H, K, h, g)$$

the set of flow lines from x_0 to x_1 with m cascades, and by $\widehat{\mathcal{M}}_m^{a,b}(x_0, x_1; H, K, h, g)$ the quotient of $\mathcal{M}_m^{a,b}(x_0, x_1; H, K, h, g)$ by the free action of \mathbb{R}^m given by s -translations on each w_j .

We also define the set of *flow lines from x_0 to x_1 with zero cascades*, denoted by

$$\mathcal{M}_0^{a,b}(x_0, x_1; H, K, h, g),$$

is the intersection $W^u(x_0; -\nabla h) \cap W^s(x_1; -\nabla h)$ whenever $\sigma_0 = \sigma_1$, and empty if σ_0 and σ_1 are different. When $\sigma_0 = \sigma_1, x_0 \neq x_1$ the quotient of $\mathcal{M}_0^{a,b}(x_0, x_1; H, K, h, g)$ by the \mathbb{R} -action is denoted by $\widehat{\mathcal{M}}_0^{a,b}(x_0, x_1; H, K, h, g)$, otherwise $\widehat{\mathcal{M}}_0^{a,b}(x_0, x_1; H, K, h, g)$ is defined to be empty.

We call

$$\widehat{\mathcal{M}}^{a,b}(x_0, x_1; H, K, h, g) = \bigcup_{m \geq 0} \widehat{\mathcal{M}}_m^{a,b}(x_0, x_1; H, K, h, g)$$

the moduli space of *Morse-Bott trajectories with cascades*. For each $m \geq 0$, the moduli space of trajectories $\widehat{\mathcal{M}}_m^{a,b}(x_0, x_1; H, K, h, g)$ with m cascades can be compactified by adding broken trajectories in the same way as Theorem A.10 in [20]. The rest of this subsection is mainly devoted to prove

Theorem 5.2. *Given two components of $\text{Crit}^{[a,b]}(\mathcal{A}_H)$, σ_0, σ_1 , and $x_i \in \sigma_i, i = 0, 1$, for generic choice of K , the set $\widehat{\mathcal{M}}^{a,b}(x_0, x_1; H, K, h, g)$ has the structure of a finite dimensional manifold and its dimension is given by*

$$\dim \widehat{\mathcal{M}}^{a,b}(x_0, x_1; H, K, h, g) = \nu(x_0) - \nu(x_1) - 1.$$

The idea of the proof of the theorem is owned to Frauenfelder [20, Theorem A.11]. By (4.3), we can write $\text{Hess} \mathcal{A}_H(z, \lambda) = T_1 + T_2$, where

$$T_1 = \mathcal{D}_s L \oplus Id_{\mathbb{R}}, \quad (T_1)^2 = Id_{\mathcal{E}},$$

and

$$T_2 = \begin{pmatrix} -\lambda \mathcal{D}_s H_{zz}(x, z) & -\mathcal{D}_s H_z(x, z) \\ -(\mathcal{D}_s H_z(x, z))^* & -1 \end{pmatrix}$$

is compact on \mathcal{E} . Since $(T_1 + T_2)^2 = Id_{\mathcal{E}} + T_1 T_2 + T_2 T_1 + T_2^2$, and $T_1 T_2 + T_2 T_1 + T_2^2$ is a compact self-adjoint operator, by the spectral theory of compact operator we obtain that the spectrum of $(T_1 + T_2)^2$ can be described as a sequence of discrete points tending to 1, and thus $\text{Spec}(\text{Hess} \mathcal{A}_H(z, \lambda)) \setminus \{0\}$ is away from 0. Let d be a constant such that

$$0 < d < \min\{|\lambda| : \lambda \in \text{Spec}(\text{Hess}(\mathcal{A}_H)(w)) \setminus \{0\}, w \in \text{Crit}^{[a,b]}(\mathcal{A}_H)\}, \quad (5.2)$$

and let $s_d : \mathbb{R} \rightarrow \mathbb{R}$ be the weight function defined by

$$s_d(t) = e^{d\vartheta(t)t},$$

where $\vartheta(t)$ is a smooth cutoff function satisfying $\vartheta(t) = -1$ for $t < 0$ and $\vartheta(t) = 1$ for $t > 1$. We introduce suitable weighted Sobolev spaces as follow:

$$\begin{aligned} L_d^2(\mathbb{R}, \mathcal{E}) &:= \{w \in L^2(\mathbb{R}, \mathcal{E}) \mid s_d w \in L^2(\mathbb{R}, \mathcal{E})\}, \\ W_d^{1,2}(\mathbb{R}, \mathcal{E}) &:= \{w \in W^{1,2}(\mathbb{R}, \mathcal{E}) \mid s_d w \in W^{1,2}(\mathbb{R}, \mathcal{E})\} \end{aligned}$$

with weighted norms

$$\begin{aligned}\|w\|_d &:= \|s_d w\|_{L^2(\mathbb{R}, \mathcal{E})}, \quad w \in L_d^2(\mathbb{R}, \mathcal{E}), \\ \|w\|_{1,d}^2 &:= \|s_d w\|_{L^2(\mathbb{R}, \mathcal{E})}^2 + \|s_d w'\|_{L^2(\mathbb{R}, \mathcal{E})}^2, \quad w \in W_d^{1,2}(\mathbb{R}, \mathcal{E}).\end{aligned}$$

We define

$$\mathcal{X} = \mathcal{E}_d^{a,b}(H, h)$$

as the Banach manifold consisting of all tuples

$$w = ((w_j)_{1 \leq j \leq m}, (t_j)_{1 \leq j \leq m-1}) \in (W_{loc}^{1,p}(\mathbb{R}, \mathcal{E}))^m \times (\mathbb{R}_+ \setminus \{0\})^{m-1}$$

with the following properties:

(i) For $1 \leq j \leq m$ there exist $p_j, q_j \in \text{Crit}^{[a,b]}(\mathcal{A}_H)$ such that

$$w_j(-\infty) = p_j, \quad w_j(+\infty) = q_j, \quad w_j \in \bar{w}_j + W_d^{1,2}(\mathbb{R}, \mathcal{E}),$$

where $\bar{w}_j : \mathbb{R} \rightarrow \mathcal{E}$ is a smooth map which for some $R \in \mathbb{R}$ satisfies $\bar{w}_j(t) \equiv p_j$ for $t \leq -R$ and $\bar{w}_j(t) \equiv q_j$ for $t \geq R$.

(ii) $p_{j+1} = \Psi_h(t_j)q_j$ for $1 \leq j \leq m-1$.

There exist two natural evaluation maps

$$\begin{aligned}\text{ev}_1 : \mathcal{X} &\rightarrow \text{Crit}^{[a,b]}(\mathcal{A}_H), \quad w \mapsto w_1(-\infty) = p_1, \\ \text{ev}_2 : \mathcal{X} &\rightarrow \text{Crit}^{[a,b]}(\mathcal{A}_H), \quad w \mapsto w_m(+\infty) = q_m.\end{aligned}$$

The tangent space of \mathcal{X} at w can be identified with a subspace of

$$\bigoplus_{j=1}^m (W_d^{1,2}(\mathbb{R}, \mathcal{E}) \times T_{p_j} \text{Crit}^{[a,b]}(\mathcal{A}_H) \times T_{q_j} \text{Crit}^{[a,b]}(\mathcal{A}_H)) \times \mathbb{R}^{m-1}$$

consisting of tuples $\omega = ((\xi_j, \zeta_{j,1}, \zeta_{j,2})_{1 \leq j \leq m}, (\tau_j)_{1 \leq j \leq m-1})$ which satisfy

$$d\Psi_h(t_j)\zeta_{j,2} + \frac{d}{dt}(\Psi_h(0)q_j)\tau_j = \zeta_{j+1,1}.$$

$T_w \mathcal{X}$ is a Banach space with norm

$$\|\omega\| := \sum_{j=1}^m (\|\xi_j\|_{1,d} + |\zeta_{j,1}|_g + |\zeta_{j,2}|_g) + \sum_{j=1}^{m-1} |\tau_j|,$$

where $|\cdot|_g$ is the norm with respect to the Riemannian metric g on $\text{Crit}(\mathcal{A}_H)$. Consider the map

$$\mathfrak{F}_K : \mathcal{X} \rightarrow \bigoplus_{j=1}^m L_d^2(\mathbb{R}, \mathcal{E}), \quad w \mapsto (w'_j(t) + \nabla^K \mathcal{A}_H(w_j(t)))_{1 \leq j \leq m}.$$

Let $\mathcal{M}_K = \mathfrak{F}_K^{-1}(0)$. The differential of \mathfrak{F}_K at $w \in \mathcal{M}_K$ is denoted by

$$d_w \mathfrak{F}_K : T_w \mathcal{X} \rightarrow \bigoplus_{j=1}^m L_d^2(\mathbb{R}, \mathcal{E}).$$

5.2.2. Proof of Theorem 5.2. We prove the theorem in three steps.

Step 1. Let $m = m(w)$ be the number of cascades. Then $d_w \mathfrak{F}_K$ is a Fredholm operator of index

$$\text{ind}(d_w \mathfrak{F}_K) = i_{\text{rel}}(\text{ev}_1(w)) + \dim_{\text{ev}_1(w)} \text{Crit}^{[a,b]}(\mathcal{A}_H) - i_{\text{rel}}(\text{ev}_2(w)) + m - 1. \quad (5.3)$$

For $1 \leq j \leq m$ denote by

$$d_{w,j} \mathfrak{F}_K : W_d^{1,2}(\mathbb{R}, \mathcal{E}) \rightarrow L_d^2(\mathbb{R}, \mathcal{E})$$

the restriction of $d_w \mathfrak{F}_K$ to

$$W_d^{1,2}(\mathbb{R}, \mathcal{E}) \equiv W_d^{1,2}(\mathbb{R}, \mathcal{E}) \times \{0\} \times \{0\} \subset W_d^{1,2}(\mathbb{R}, \mathcal{E}) \times T_{p_j} \text{Crit}^{[a,b]}(\mathcal{A}_H) \times T_{q_j} \text{Crit}^{[a,b]}(\mathcal{A}_H),$$

which is a Fredholm operator given by

$$d_{w,j} \mathfrak{F}_K = \frac{d}{dt} + K(w_j(t)) \text{Hess} \mathcal{A}_H(w_j(t)) + \{dK(w_j(t))[\cdot]\} \nabla \mathcal{A}_H(w_j(t)).$$

By deforming $K(\cdot)$ linearly to zero as in Proposition 4.9, we see that

$$\text{ind}(d_{w,j} \mathfrak{F}_K) = \text{ind}(L_{w,j}), \quad (5.4)$$

where $L_{w,j} : W_d^{1,2}(\mathbb{R}, \mathcal{E}) \rightarrow L_d^2(\mathbb{R}, \mathcal{E})$ is defined by

$$L_{w,j} = \frac{d}{dt} + \text{Hess} \mathcal{A}_H(w_j(t)).$$

By conjugating $L_{w,j}$ with s_d we define the operator

$$\tilde{L}_{w,j} = s_d L_{w,j} s_{-d} : W^{1,2}(\mathbb{R}, \mathcal{E}) \rightarrow L^2(\mathbb{R}, \mathcal{E}).$$

This is also Fredholm and satisfies

$$\text{ind}(\tilde{L}_{w,j}) = \text{ind}(L_{w,j}). \quad (5.5)$$

For $\xi \in W^{1,2}(\mathbb{R}, \mathcal{E})$ we calculate

$$\begin{aligned} \tilde{L}_{w,j} \xi &= s_d L_{w,j} (s_{-d} \xi) \\ &= \frac{d\xi}{dt} + (\text{Hess} \mathcal{A}_H(w_j(t)) - d(\vartheta(t) + \vartheta'(t)t) \text{id}) \xi. \end{aligned}$$

Set

$$A_j(t) := \text{Hess} \mathcal{A}_H(w_j(t)) - d(\vartheta(t) + \vartheta'(t)t) \text{id}.$$

Then

$$A_j(-\infty) = \text{Hess}\mathcal{A}_H(p_j) + d \text{ id} \quad \text{and} \quad A_j(+\infty) = \text{Hess}\mathcal{A}_H(q_j) - d \text{ id}.$$

The inequality (5.2) implies that $A_j(\pm\infty)$ are invertible and satisfy

$$\begin{aligned} V^-(A_j(-\infty)) &= V^-(\text{Hess}\mathcal{A}_H(p_j)) \quad \text{and} \\ V^-(A_j(+\infty)) &= V^-(\text{Hess}\mathcal{A}_H(q_j)) \oplus \ker(\text{Hess}\mathcal{A}_H(q_j)). \end{aligned}$$

It follows from Lemma 4.8 that $\tilde{L}_{w,j}$ is a Fredholm operator of index

$$\begin{aligned} \text{ind}(\tilde{L}_{w,j}) &= \dim(V^-(A_j(-\infty)), V^-(A_j(+\infty))) \\ &= \dim(V^-(A_j(-\infty)), V^-(\text{Hess}\mathcal{A}_H(q_j))) \\ &\quad + \dim(V^-(\text{Hess}\mathcal{A}_H(q_j)), V^-(A_j(+\infty))) \\ &= \dim(V^-(\text{Hess}\mathcal{A}_H(p_j)), V^-(\text{Hess}\mathcal{A}_H(q_j))) \\ &\quad + \dim(V^-(\text{Hess}\mathcal{A}_H(q_j)), V^-(\text{Hess}\mathcal{A}_H(q_j)) \oplus \ker(\text{Hess}\mathcal{A}_H(q_j))) \\ &= i_{\text{rel}}(p_j) - i_{\text{rel}}(q_j) - \dim_{q_j} \text{Crit}^{[a,b]}(\mathcal{A}_H). \end{aligned} \tag{5.6}$$

Combining the following index formulation

$$\text{ind}(d_w \mathfrak{F}_K) = \sum_{j=1}^m \text{ind}(d_{w,j} \mathfrak{F}_K) + \dim_{p_1} \text{Crit}^{[a,b]}(\mathcal{A}_H) + \sum_{j=1}^m \dim_{q_j} \text{Crit}^{[a,b]}(\mathcal{A}_H) + m - 1$$

with (5.4), (5.5) and (5.6), we obtain (5.3).

Step 2. For generic $K \in \mathbf{K}$ the set $\mathcal{M}_K = \mathfrak{F}_K^{-1}(0)$ is a finite dimensional manifold with the local dimension $\dim_w \mathcal{M}_K = \text{ind} \mathfrak{F}_K$.

We define

$$\mathfrak{F} : \mathcal{X} \times \mathbf{K} \rightarrow \bigoplus_{j=1}^m L_d^2(\mathbb{R}, \mathcal{E}), \quad (w, K) \mapsto \mathfrak{F}_K(w),$$

and set

$$\Theta := \{(w, K) \in \mathcal{X} \times \mathbf{K} \mid \mathfrak{F}(w, K) = 0\}.$$

For each $(w, K) \in \Theta$, the derivative of \mathfrak{F} at (w, K) is given by

$$d\mathfrak{F}(w, K)(\zeta, \kappa) = d\mathfrak{F}_K(w)\zeta + (\kappa(w_j)\nabla\mathcal{A}_H(w_j))_{1 \leq j \leq m}, \quad \forall (\zeta, \kappa) \in T_{(w,K)}(\mathcal{X} \times \mathbf{K}).$$

We claim that $d\mathfrak{F}(w, K)$ is surjective. In fact, since $d\mathfrak{F}_K(w)$ is a Fredholm operator, it and hence $d\mathfrak{F}(w, K)$ has a closed range and a finite dimensional cokernel. So it suffices to prove that $\text{Im}(d\mathfrak{F}(w, K))$ is dense. By a contradiction we assume that $(\text{Im}(d\mathfrak{F}(w, K)))^\perp$ contains a nonzero element

$$\eta = (\eta_j)_{1 \leq j \leq m} \in \bigoplus_{j=1}^m L_d^2(\mathbb{R}, \mathcal{E}).$$

Then we have

$$\sum_{j=1}^m \int_M \langle \eta_j, (d\mathfrak{F}_K(w)\zeta)_j \rangle dt = 0, \quad \forall \zeta \in T_w \mathcal{X}, \quad (5.7)$$

$$\sum_{j=1}^m \int_M \langle \eta_j, \kappa(w_j) \nabla \mathcal{A}_H(w_j) \rangle dt = 0, \quad \forall \kappa \in \mathbf{K}. \quad (5.8)$$

(5.7) implies that for all $\xi_j \in W^{1,2}(\mathbb{R}, \mathcal{E})$,

$$\int_M \left\langle \eta_j, \frac{d}{dt} \xi_j(t) + K(w_j(t)) \text{Hess} \mathcal{A}_H(w_j(t)) \xi_j(t) + \{dK(w_j(t))[\xi_j(t)]\} \nabla \mathcal{A}_H(w_j(t)) \right\rangle dt = 0.$$

It follows that η_j is C^1 for $1 \leq j \leq m$. These and (5.8) yield that η vanishes identically. (See Lemma 6.2 for a similar proof). This contradiction leads to the claim. Hence it follows from the implicit function theorem that Θ is a Banach manifold.

Consider the projection

$$\pi : \Theta \rightarrow \mathbf{K}, \quad (w, K) \rightarrow K,$$

which has the differential

$$d\pi(w, K) : T_{(w, K)} \Theta \rightarrow T_K \mathbf{K}, \quad (\zeta, \kappa) \rightarrow \kappa.$$

The kernel of $d\pi(w, K)$ is isomorphic to the kernel of $d\mathfrak{F}_K(w)$. The fact that $d\mathfrak{F}(w, K)$ is surjective implies that $d\pi(w, K)$ has the same codimension as the image of $d\mathfrak{F}_K(w)$. Thus $d\pi(w, K)$ is a Fredholm operator of the same index as $d\mathfrak{F}(w, K)$. It follows from the Sard-Smale Theorem that all regular values of π forms a residual (and thus dense) subset in \mathbf{K} . And such regular values correspond to K for which $d\mathfrak{F}_K(w)$ is surjective for each $w \in \mathfrak{F}_K^{-1}(0)$.

Step 3. By step 2 for generic $K \in \mathbf{K}$ the set $\widehat{\mathcal{M}}_m^{a,b}(x_0, x_1; H, K, h, g)$ can be endowed with the structure of a manifold with corners of dimension $\nu(x_0) - \nu(x_1) - 1$. Set

$$\mathcal{M}_{\leq l}(x_0, x_1) := \bigcup_{1 \leq m \leq l} \widehat{\mathcal{M}}_m^{a,b}(x_0, x_1; H, K, h, g), \quad \forall l \in \mathbb{N}.$$

We show by induction on l that for generic $K \in \mathbf{K}$ the set $\mathcal{M}_{\leq l}(x_0, x_1)$ has the structure of manifold of dimension $\nu(x_0) - \nu(x_1) - 1$. For $l = 1$ this is clear since $\mathcal{M}_{\leq 1}(x_0, x_1) = \widehat{\mathcal{M}}_1^{a,b}(x_0, x_1; H, K, h, g)$. As we said above Theorem 5.2, $\mathcal{M}_{\leq l}(x_0, x_1)$ can be compactified to a manifold with corners $\bar{\mathcal{M}}_{\leq l}(x_0, x_1)$ such that

$$\partial \bar{\mathcal{M}}_{\leq l}(x_0, x_1) = \partial \widehat{\mathcal{M}}_{l+1}^{a,b}(x_0, x_1; H, K, h, g).$$

So $\mathcal{M}_{\leq l+1}(x_0, x_1) = \mathcal{M}_{\leq l}(x_0, x_1) \cup \widehat{\mathcal{M}}_{l+1}^{a,b}(x_0, x_1; H, K, h, g)$ has a finite dimensional manifold structure with

$$\dim \mathcal{M}_{\leq l+1}(x_0, x_1) = \dim \mathcal{M}_{\leq l}(x_0, x_1) = \nu(x_0) - \nu(x_1) - 1.$$

□

Assume that $\nu(x_0) = \nu(x_1) + 1$. Then the set $\widehat{\mathcal{M}}^{a,b}(x_0, x_1; H, K, h, g)$ is finite by Theorem 5.2. Put

$$n(x_0, x_1) = \#\widehat{\mathcal{M}}^{a,b}(x_0, x_1; H, K, h, g).$$

The chain group $BC_*^{[a,b]}(H, K, h, g)$ is defined as the finite dimension \mathbb{Z}_2 -vector space given by

$$BC_*^{[a,b]}(H, K, h, g) := \text{Crit}^{[a,b]}(h) \otimes \mathbb{Z}_2,$$

where $\text{Crit}^{[a,b]}(h) := \text{Crit}(h) \cap \text{Crit}^{[a,b]}(\mathcal{A}_H)$. The grading is given by the above index ν and the differential operator is defined by

$$\partial x = \sum_{\substack{y \in \text{Crit}^{[a,b]}(h), \\ \nu(y) = \nu(x) - 1}} (n(x, y) \bmod 2) y.$$

Using the compactness in Proposition 3.5 and a standard gluing construction as in [20], we can prove that $\partial^2 = 0$. Thus $(BC_*^{[a,b]}(H, K, h, g), \partial)$ is a chain complex, still called the *Rabinowitz Floer complex* of \mathcal{A}_H . The corresponding homology

$$HF_*^{[a,b]}(H, K, h, g) := H_*(BC_*^{[a,b]}(H, K, h, g), \partial)$$

is called the *Rabinowitz Floer homology* of \mathcal{A}_H . Standard arguments show that $HF_*^{[a,b]}(H, K, h, g)$ is independent up to canonical isomorphism of the choices of H, K, h and g , see [20, Frauenfelder] for details. So $HF_*^{[a,b]}(H, K, h, g)$ will be simply denoted by $HF_*^{[a,b]}(H)$.

5.3 Continuation of the Rabinowitz-Floer homology

In this subsection, we show that under a small perturbation of the pair (H, K) there exists a natural isomorphism between the Rabinowitz-Floer homology of \mathcal{A}_H and that of the perturbed functional $\mathcal{A}_{\tilde{H}}$. Then by taking a partition of a smooth path connecting from (H_0, K_0) to (H_1, K_1) , under suitable hypotheses we prove $RHF_k^{[a,b]}(H_0, K_0) = RHF_k^{[a,b]}(H_1, K_1)$. In order to control length of this paper we only consider the Morse situation. The similar proof can be completed in the Morse-Bott situation; see [20, Appendix A].

Fix a constant ϵ as in Proposition 3.4. Assume that $(H_0, K_0), (H_1, K_1) \in \Omega_{reg}$ and

$$\sup_{z \in E_s} \int_M |H_0(x, z) - H_1(x, z)| dx + \|K_0 - K_1\|_{\mathbf{K}} < \frac{\epsilon}{10}. \quad (5.9)$$

Let $\beta(t) \in C^\infty(\mathbb{R}, [0, 1])$ satisfy $\beta(t) \equiv 0$ for $t \leq 0$, $\beta(t) \equiv 1$ for $t \geq 1$, and $0 \leq \beta'(t) \leq 2$ for all t . We define the t -dependent functions $H(t, x, z)$, $K(t, w)$ by

$$H(t, x, z) = (1 - \beta(t))H_0(x, z) + \beta(t)H_1(x, z), \quad K(t, w) = (1 - \beta(t))K_0(w) + \beta(t)K_1(w).$$

It is not hard to check that they satisfy the assumptions (i)-(iii) with $A = \epsilon/5$ at the beginning of Section 3.3. By replacing (H, K) by an arbitrary small perturbation we can assume that (H, K) is regular in the sense that the map $\mathcal{F}_{H,K} : Q_1 \rightarrow Q_0$ given by

$$\mathcal{F}_{H,K}(w)(t) = \frac{dw(t)}{dt} + (I + K(t, w(t)))\nabla \mathcal{A}_H(t, x, w(t))$$

is transversal to $0 \in Q_1$, where Q_0 and Q_1 are as in (4.5). Hence for given any pair of critical points $w_0 \in \text{Crit}(\mathcal{A}_{H_0})$ and $w_1 \in \text{Crit}(\mathcal{A}_{H_1})$, if $w(t)$ is a solution of (3.17) satisfying $w(-\infty) = w_0$ and $w(+\infty) = w_1$, by Proposition 3.4 $w(t)$ is uniformly bounded by a constant depending only on w_0 and w_1 . This uniform boundedness implies precompactness, therefore we can define the moduli space of trajectories of the negative non-autonomous gradient flow

$$\overline{\mathcal{M}}(w_0, w_1) = \left\{ w \in C^1(\mathbb{R}, \mathcal{E}) \left| \begin{array}{l} w(t) \text{ is a solution of (3.17) with} \\ w(-\infty) = w_0 \text{ and } w(+\infty) = w_1 \end{array} \right. \right\}.$$

One can show that $\overline{\mathcal{M}}(w_0, w_1)$ is either empty or a manifold of dimension

$$\dim \overline{\mathcal{M}}(w_0, w_1) = i_{\text{rel}}^{H_0}(w_0) - i_{\text{rel}}^{H_1}(w_1),$$

where $i_{\text{rel}}^{H_j}(w_j)$ is the relative index with respect to H_j at w_j , $j = 0, 1$. The key techniques of the proof are compactification of broken trajectories and the gluing construction very similar to that of the autonomous case, which we will not reproduce here. If $i_{\text{rel}}^{H_0}(w_0) = i_{\text{rel}}^{H_1}(w_1)$, then the integer $n(w_0, w_1) := \# \overline{\mathcal{M}}(w_0, w_1)$ is finite. For each $k \in \mathbb{Z}$, we consider a homomorphism

$$\Psi_{01} : \text{CF}_k(H_0) \rightarrow \text{CF}_k(H_1)$$

defined by

$$\Psi_{01} \left(\sum_{j=1}^l m_j x_j \right) = \sum_{j=1}^l \sum_{y \in \text{Crit}_k(\mathcal{A}_{H_1})} (n(x_j, y) \bmod [2]) m_j y$$

for $\sum_{j=1}^l m_j x_j \in \text{CF}_k(H_0)$ with $x_j \in \text{Crit}(\mathcal{A}_{H_0})$ and $m_j \in \mathbb{Z}_2$, $j = 1, \dots, m$. We claim: Ψ_{01} is a chain homomorphism. That is, the following diagram communicates

$$\begin{array}{ccc} \text{CF}_{k+1}(H_0) & \xrightarrow{\partial_{k+1}^0} & \text{CF}_k(H_0) \\ \downarrow \Psi_{01} & & \downarrow \Psi_{01} \\ \text{CF}_{k+1}(H_1) & \xrightarrow{\partial_k^1} & \text{CF}_k(H_1), \end{array}$$

where ∂^j , $j = 0, 1$, are boundary operators corresponding to (H_j, K_j) . The proof is standard. In fact, we only need to consider the 1-dimension moduli space $\overline{\mathcal{M}}(w_0, w_1)$ with $i_{\text{rel}}^{H_0}(w_0) = i_{\text{rel}}^{H_1}(w_1) + 1$. The boundary of $\overline{\mathcal{M}}(w_0, w_1)$ then splits into two parts:

$$\begin{aligned} \partial \overline{\mathcal{M}}(w_0, w_1) = & \left(\bigcup_{x \in \text{Crit}_{k+1}(\mathcal{A}_{H_1})} \overline{\mathcal{M}}(w_0, x) \times \mathcal{M}_{H_1, K_1}(x, w_1) \right) \\ & \bigcup \left(\bigcup_{y \in \text{Crit}_k(\mathcal{A}_{H_0})} \mathcal{M}_{H_0, K_0}(w_0, y) \times \overline{\mathcal{M}}(y, w_1) \right). \end{aligned}$$

The first part appears in $\partial^0 \circ \Psi_{01}$ while the second parts does in $\Psi_{01} \circ \partial^1$. The desired claim follows.

Moreover, we have

Lemma 5.3. *If $(H_l, K_l) \in \Omega_{reg}$, $l = 1, 2, 3$, satisfy*

$$\sup_{z \in E_s} \int_M |H_m(x, z) - H_n(x, z)| dx + \|K_0 - K_1\|_{\mathbf{K}} < \frac{\epsilon}{5} \quad \forall m, n \in \{1, 2, 3\},$$

then $\Psi_{ln} = \Psi_{lm} \circ \Psi_{mn}$ and $\Psi_{ll} = id$. In particular, Ψ_{12} is an isomorphism.

Again using the result of compactness and the standard arguments in [7, 17, 39], we arrive at the conclusion of the above lemma. The proof is omitted here. By transversality, for each H satisfying (H1) – (H4), there exists $(\tilde{H}, \tilde{K}) \in \Omega_{reg}$ such that $|H - \tilde{H}|$ is small enough, then one can define the Rabinowitz-Floer homology of \mathcal{A}_H to be that of $\mathcal{A}_{\tilde{H}}$.

Proposition 5.4 (Global continuation). *If $(H_0, K_0), (H_1, K_1) \in \Omega_{reg}$, then it holds that*

$$RHF_*^{[a,b]}(H_0, K_0) = RHF_*^{[a,b]}(H_1, K_1).$$

Proof. We prove the propostion in two steps.

Step 1. Let us make an additional hypothesis that $\sup_{z \in E_s} \int_M |H_0(x, z) - H_1(x, z)| dx$ is finite. Let ϵ be as in (5.9). Given a smooth path (H_s, K_s) with $s \in [0, 1]$ from (H_0, K_0) to (H_1, K_1) , for example one can choose $H_s(x, z) = sH_0(x, z) + (1 - s)H_1(x, z)$ and then take a partition $0 = s_0 < s_1 < \dots < s_m = 1$ such that

$$\sup_{z \in E_s} \int_M |H_{s_{j+1}}(x, z) - H_{s_j}(x, z)| dx < \frac{\epsilon}{5}, \quad \|K_{s_{j+1}} - K_{s_j}\|_{\mathbf{K}} < \frac{\epsilon}{5}, \quad j \in \{0, \dots, m\}.$$

It follows from Lemma 5.3 that there exist isomorphisms

$$\Psi_{j,j+1} : RHF_*(H_{s_j}, K_{s_j}) = RHF_*(H_{s_{j+1}}, K_{s_{j+1}}).$$

By composing these isomorphisms we get an isomorphism between the Rabinowitz-Floer homologies of \mathcal{A}_{H_0} and \mathcal{A}_{H_1} .

Step 2. If $(z, \lambda) \in \text{Crit}_k^{[a,b]}(\mathcal{A}_{H_l})$ then $z \in \Sigma_1(H_l) = \{z \in E_s \mid \int_M H(x, z(x)) dx \leq 1\}$, $l = 0, 1$. From the proof of Proposition 3.1, we see that the assumption (H4) implies that $\Sigma_1(H_0)$ and $\Sigma_1(H_1)$ are bounded sets in E_s . It follows from Proposition 3.3 that the negative gradient flow lines connecting two critical points are uniformly bounded in \mathcal{E} . Take a ball $B_R(0) \subset E_s$ such that $\Sigma_1(H_l)$ and the z -components of corresponding connecting orbits are all contained within it. For $\delta > 0$, one chooses a smooth function $\chi_\delta(t)$ such that $\chi_\delta(t) = 1$ for $0 \leq t \leq R$ and $\chi_\delta(t) = 0$ for $t \geq (R + \delta)$. Given $K^\delta \in \mathbf{K}$, consider the modified function on $\Sigma M \oplus \Sigma M$,

$$H^\delta(x, z) = \chi_\delta(\|z\|)H_0(x, z) + (1 - \chi_\delta(\|z\|))H_1(x, z).$$

Clearly, $\sup_{z \in E_s} \int_M |H^\delta(x, z) - H_1(x, z)| dx < +\infty$. By step 1, it holds

$$RHF_*^{[a,b]}(H^\delta, K^\delta) = RHF_*^{[a,b]}(H_1, K_1) \quad (5.10)$$

Moreover, since $\Sigma_1(H^\delta) \rightarrow \Sigma_1(H_0)$ as $\delta \rightarrow 0$, and the Rabinowitz-Floer groups are defined in terms of the critical points and connecting orbits between them, we obtain

$$RHF_*^{[a,b]}(H^\delta, K^\delta) = RHF_*^{[a,b]}(H_0, K_0). \quad (5.11)$$

The desired result follows from (5.10) and (5.11). \square

Because of the above theorem we can simply write $RHF_k^{[a,b]}(H) = RHF_k^{[a,b]}(H, K)$. As we said before, the global continuation also holds in the Morse-Bott situation. In particular, if \mathcal{A}_H is Morse, we can take h vanishing identically on $\text{Crit}^{[a,b]}(\mathcal{A}_H)$, and obtain $RHF_*^{[a,b]}(H) = HF_*^{[a,b]}(H)$.

6 Transversality

In this section we first show that the nonlinearity H can be slightly perturbed so that \mathcal{A}_H is Morse. Then following the ideas of Abbondandolo and Majer [3] we can make a small perturbation of \mathcal{A}_H such that the perturbed functional $\tilde{\mathcal{A}}$ satisfies the Morse-Smale condition.

Consider the Gevrey space \mathbf{G} of C^∞ functions $h : \Sigma M \oplus \Sigma M \rightarrow \mathbb{R}$ with the norm

$$|h|_{\mathbf{G}} := \sup_{k \in \mathbb{N}} \frac{\|h\|_{C^k}}{(k!)^4} < +\infty.$$

Here the $\|\cdot\|_{C^k}$ -norm can be explicitly given as follows: Because of compactness of M we choose a finite open cover $\{\mathcal{U}_k\}_{k=1}^m$ of M consisting of domains of chart maps $\varphi_k : \mathcal{U}_k \rightarrow \Omega_k$, $k = 1, \dots, m$, where $\Omega_k = \varphi_k(\mathcal{U}_k) \subset \mathbb{R}^n$ is an open unit ball; and therefore there exist local trivializations $\Phi_k : \Sigma M \oplus \Sigma M|_{\mathcal{U}_k} \rightarrow \mathcal{U}_k \times \Sigma^n \times \Sigma^n$, $k = 1, \dots, m$. Fix a partition $\{\lambda_k\}_{k=1}^m$ of unity subordinate to $\{\mathcal{U}_k\}_{k=1}^m$, and define for any smooth function $h : \Sigma M \oplus \Sigma M \rightarrow \mathbb{R}$,

$$\|h\|_{C^k} = \sum_k \sup_{\bar{\Omega}_k \times (\Sigma^n)^2} \left\| d^k((\lambda_k \circ \varphi_k^{-1}) \cdot (h \circ \Phi_k^{-1}))(x, u, v) \right\|.$$

It is not hard to see that any alternative choice of finite open covering of charts, trivializations and partition of unity gives an equivalent norm on \mathbf{G} . Let us fix such choice. Then \mathbf{G} is a separable Banach space [26].

Lemma 6.1. *Assume that $H \in C^2(\Sigma M \oplus \Sigma M)$ satisfies (H1) – (H4). Then \mathcal{A}_{H+h} is a Morse function for a generic perturbation h of H in \mathbf{G} .*

Proof. We define a map $\Psi : \mathcal{E} \times \mathbf{G} \rightarrow \mathcal{E}$ by

$$\Psi(w, h) = \nabla \mathcal{A}_{H+h}(w), \quad (6.1)$$

where $w = (z, \lambda) \in \mathcal{E}$. One can easily check that Ψ is a map of class C^1 .

We first prove that 0 is a regular value of Ψ . Since we have assume $0 \notin \text{spec}(D)$, for each $(w, h) \in \Psi^{-1}(0)$ with $w = (z, \lambda)$ it must hold that $\lambda \neq 0$. It is not hard to see that the derivative of Ψ at $(w, h) \in \Psi^{-1}(0)$ with respect to w is given by

$$d_w \Psi(w, h) = \begin{pmatrix} \mathcal{D}_s L & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\lambda \mathcal{D}_s(H_{zz} + h_{zz}) & -\mathcal{D}_s(H_z + h_z) \\ -(\mathcal{D}_s(H_z + h_z))^* & -1 \end{pmatrix}.$$

This is Fredholm operator with Fredholm index 0 since it is a compact perturbation of the invertible operator $\begin{pmatrix} \mathcal{D}_s L & 0 \\ 0 & 1 \end{pmatrix}$. Hence the range of $d_w \Psi(w, h)$ has finite codimension. To prove that $d\Psi(w, h)$ is surjective, we only need to show that the range of the derivative of Ψ with respect to h at (w, h) given by

$$d_h \Psi(w, h)X = \begin{pmatrix} -\lambda \mathcal{D}_s X_z(x, z) \\ -\int_M X(x, z) dx \end{pmatrix} \quad (6.2)$$

is dense in \mathcal{E} , where $X \in T_h \mathbf{G} = \mathbf{G}$. Choosing $X \equiv \text{constant} \neq 0$ and substituting it into (6.2), we see that the second component of $d_h \Psi(w, h)X$ spans \mathbb{R} .

Claim. *The first component of $d_h \Psi(w, h)$ has dense range in E_s .*

In fact, consider the element of $C^1(M, \Sigma M \oplus \Sigma M)$ of form $(a\phi, b\varphi)$, where $(\phi, \varphi) \in C^1(M, \Sigma M \oplus \Sigma M)$ and $a, b \in C^\infty(M)$ satisfy the following condition

$$\sup_{x \in M} \{ |d^j a(x)|, |d^j b(x)| \} \leq C(j!)^4 \quad \forall j \in \mathbb{N} \cup \{0\}$$

for some constant C . Define the function $\bar{X} : \Sigma M \oplus \Sigma M \rightarrow \mathbb{R}$ by

$$\bar{X}(x, u, v) = a(x)\langle \phi(x), u \rangle + b(x)\langle \varphi(x), v \rangle.$$

Then $\bar{X} \in \mathbf{G}$ and

$$\bar{X}_z(x, z(x)) = (a(x)\phi(x), b(x)\varphi(x))^T.$$

Denote by Δ the set consisting of all such \bar{X} . Since $\{\bar{X}_z \mid \bar{X} \in \Delta\}$ is dense in $C^1(M, \Sigma M) \times C^1(M, \Sigma M)$, and \mathcal{D}_s maps this set into a dense subspace in E_s , we deduce that the set

$$\{\text{the first component of } d_h \Psi(w, h)\bar{X} \mid \bar{X} \in \Delta\} = \{-\lambda \mathcal{D}_s X_z \mid \bar{X}_z \in \Delta\}$$

is dense in E_s . Here we use the fact that $\lambda \neq 0$, which comes from the assumption that $0 \notin \text{Spect}(D)$ as showed at the beginning. Hence 0 is a regular value of Ψ .

Next we consider the C^1 -submanifold $\mathcal{Z} = \{(w, h) \in \mathcal{E} \times \mathbf{G} \mid \Psi(w, h) = 0\}$ and the projection $\pi : \mathcal{Z} \rightarrow \mathbf{G}$ given by $\pi(w, h) = h$. By a standard argument [40], the Fredholm property of $d_w \Psi(w, h)$ implies that the derivative $d\pi(w, h)$ is Fredholm and has the same index 0 as $d_w \Psi(w, h)$. Therefore all regular values of π form a residual subset of \mathbf{G} by the Sard-Smale theorem [42]. Moreover, $h \in \mathbf{G}$ is a regular value of π if and only if \mathcal{A}_{H+h} is a Morse functional. Hence \mathcal{A}_{H+h} is a Morse functional for a generic $h \in \mathbf{G}$. \square

With the same strategy as the above arguments we shall prove

Lemma 6.2. *Let $H \in C^2(\Sigma M \oplus \Sigma M)$ satisfying (H1) – (H4). Then for generic $K \in \mathbf{K}$ the map $\mathcal{F}_{H,K} : Q_1 \rightarrow Q_0$ in (4.5) satisfying $i_{\text{rel}}(w_1) - i_{\text{rel}}(w_2) = 0$ has the regular value $0 \in Q_0$.*

Proof. We divide our proof into two steps.

Step 1. Consider the map $\mathcal{F}_H : Q_1 \times \mathbf{K} \rightarrow Q_0$ defined by

$$\mathcal{F}_H(w, K) = \mathcal{F}_{H,K}(w) = \frac{dw}{dt} + \nabla^K \mathcal{A}_H(w).$$

It is of class C^1 by our assumption, and the derivative of \mathcal{F}_H at (w, K) is given by

$$d\mathcal{F}_H(w, K)(y, \kappa) = d\mathcal{F}_{H,K}(w)y + \kappa(w)\nabla \mathcal{A}_H(w), \quad \forall (y, \kappa) \in T_{(w,K)}Q_1 \times \mathbf{K}.$$

Since $d\mathcal{F}_{H,K}(w)$ is a Fredholm operator with the index $i_{\text{rel}}(w_1) - i_{\text{rel}}(w_2) = 0$, it has a closed range and a finite dimensional cokernel. Therefore, $d\mathcal{F}_H(w, K)$ has a closed range and a finite dimensional cokernel. We claim that $0 \in Q_0$ is a regular value of \mathcal{F}_H . Arguing by contradiction, assume that there exists $(w, K) \in (\mathcal{F}_H)^{-1}(0)$ such that $d\mathcal{F}_H(w, K)$ is not surjective. Then there exists $\psi \in L^2(\mathbb{R}, \mathcal{E}) \setminus \{0\}$ such that

$$\int_M \langle \psi(t), d\mathcal{F}_{H,K}(w(t))y \rangle dt = 0, \quad \forall y \in W^{1,2}(\mathbb{R}, \mathcal{E}), \quad (6.3)$$

$$\int_M \langle \psi(t), \kappa(w(t))\nabla \mathcal{A}_H(w(t)) \rangle dt = 0, \quad \forall \kappa \in \mathbf{K}. \quad (6.4)$$

(6.3) implies that $\psi(t)$ is a weak solution of the adjoint equation $(d\mathcal{F}_{H,K}(w(t)))^* \psi(t) = 0$ and thus continuous. For any $\kappa \in \mathcal{NS}(\mathcal{E}, C^2 \oplus \mathbb{R})$ and a fixed $t_0 \in \mathbb{R}$, we put

$$\kappa^\epsilon(w) = \frac{1}{\epsilon} \rho \left(\frac{\|w - w(t_0)\|_{\mathcal{E}}}{\epsilon} \right) \kappa, \quad (6.5)$$

where ρ is as in (2.19). Substituting κ^ϵ into (6.4) and taking $\epsilon \rightarrow 0$, we get

$$\frac{\int_{-1}^1 e^{-1/(1-s^2)} ds}{\|w'(t_0)\|_{\mathcal{E}}} \langle \psi(t_0), \kappa \nabla \mathcal{A}_H(w(t_0)) \rangle = 0. \quad (6.6)$$

Since $\nabla \mathcal{A}_H(w(t_0)) \neq 0$ and $C^2 \oplus \mathbb{R}$ is dense in \mathcal{E} , it follows from Proposition 2.3 that $\psi(t_0) = 0$. But t_0 is arbitrary. We arrive at $\psi = 0$, which contradicts with our assumption $\psi \neq 0$. So $d\mathcal{F}_H(w, K)$ is onto.

Step 2. Now $\mathcal{Z} := \mathcal{F}_H^{-1}(0)$ is a C^1 -submanifold of $Q_1 \times \mathbf{K}$ by Step 1, and the projection $Q_1 \times \mathbf{K} \rightarrow \mathbf{K}$ restricts to a Fredholm map $\pi : \mathcal{Z} \rightarrow \mathbf{K}$ with index 0. Hence the set of π 's regular values is of second category by the Sard-Smale theorem and satisfies the property of Lemma 6.2. \square

Remark 6.3. Lemma 6.1 and Lemma 6.2 imply that for generic $h \in \mathbf{G}$ and generic $K \in \mathbf{K}$ the functional \mathcal{A}_{H+h} has the *Morse-Smale property up to order 0*, that is to say, for each two critical points w_1, w_2 of \mathcal{A}_{H+h} such that $i_{\text{rel}}(w_1) - i_{\text{rel}}(w_2) = 0$ the unstable manifold of w_1 and the stable manifold of w_2 meet transversally. However, according to the usual method the

perturbed functional should own the Morse-Smale property up to at least order 2 for constructing the Rabinowitz-Floer complex. This requires that $\nabla \mathcal{A}_H$ should be at least of class C^3 because using the Sard-Smale theorem for Fredholm maps to obtain the Morse-Smale property requires the regularity to be strictly higher than the Fredholm index. But it is regrettable that when $\dim M \geq 3$ the functional $\mathcal{A}_H : \mathcal{E} \rightarrow \mathbb{R}$ is at most of class C^2 even if H is C^∞ . Fortunately, the functional $\mathcal{A}_H : \mathcal{E} \rightarrow \mathbb{R}$ can be written as

$$\mathcal{A}_H(z, \lambda) = \frac{1}{2} \int_M \langle Lz(x), z(x) \rangle dx + G(z, \lambda),$$

where $G(z, \lambda) := -\lambda \int_M \{H(x, z(x)) - 1\} dx$ and the gradient of the functional $G : \mathcal{E} \rightarrow \mathbb{R}$,

$$(z, \lambda) \mapsto \nabla G(z, \lambda) = \begin{pmatrix} -\lambda \mathcal{D}_s H_z(x, z) \\ -\int_M (H(x, z) - 1) dx \end{pmatrix},$$

is a compact map on \mathcal{E} by Proposition 2.1. Thus according to [3, Appendix B] (see [3, page 758] for precise explanation) we can approximate \mathcal{A}_H by a smooth functional $\tilde{\mathcal{A}}$ on \mathcal{E} with the Morse-Smale property up to every order.

Lemma 6.4. *Given h as in Lemma 6.1 and two real numbers a, b with $a < b$, for each $\epsilon > 0$ there exists a functional $\tilde{\mathcal{A}} \in C^\infty(\mathcal{E})$ such that*

- (i) $\|\mathcal{A}_{H+h} - \tilde{\mathcal{A}}\|_{C^2(\mathcal{E})} < \epsilon$,
- (ii) $\tilde{\mathcal{A}}$ satisfies the $(PS)_c$ -condition in $[a - \epsilon, b + \epsilon]$,
- (iii) $\tilde{\mathcal{A}}$ has the Morse-Smale property up to each order,
- (iv) $\tilde{\mathcal{A}}$ and \mathcal{A}_{H+h} have the same critical points and the same connecting orbits.

Remark that by Lemma 6.4 the perturbed functional $\tilde{\mathcal{A}}$ can be used to define a boundary homomorphism

$$\partial : C_k(\tilde{\mathcal{A}}, [a, b]) \longrightarrow C_{k-1}(\tilde{\mathcal{A}}, [a, b])$$

which is the same as (5.1). The invariance of the homology [3, Section 9] implies that different perturbed functionals give the same homology. The Rabinowitz-Floer homology $RHF_*(H)$ is then defined to be the homology of the above complex.

7 Existence results for the coupled Dirac system

In this section, we prove Theorem 1.1 via computing the Rabinowitz-Floer homology. To do it, we choose a special nonlinearity $H_0(x, u, v) = \frac{1}{2}(|u|^2 + |v|^2)$ whose homology can be easily computed, then by continuation we get the desired result.

7.1 Computations of Rabinowitz-Floer homology

By (2.12), each critical (z, λ) of \mathcal{A}_{H_0} satisfies

$$\begin{cases} Lz = \lambda z, \\ \int_M H_0(x, z) dx = 1. \end{cases} \quad (7.1)$$

So each connected component σ_k of $\text{Crit}(\mathcal{A}_{H_0})$ has the form

$$\{(z_k, \bar{\lambda}_k) \in \mathcal{E} \times \mathbb{R} \mid (z_k, \bar{\lambda}_k) \text{ satisfies (7.1)}\},$$

where $\bar{\lambda}_k$ is a fixed eigenvalue of L . If $\bar{\lambda}_k$ has multiplicity m_k , then σ_k is a manifold diffeomorphic to a sphere S^{2m_k-1} and has the tangent space at $(z_k, \bar{\lambda}_k)$

$$T_{(z_k, \bar{\lambda}_k)} \sigma_k = \{(z, 0) \in \mathcal{E} \times \mathbb{R} \mid Lz = \bar{\lambda}_k z, (z_k, z)_{L^2} = 0\}.$$

Since

$$\text{Hess} \mathcal{A}_{H_0}(z, \lambda) = \begin{pmatrix} \mathcal{D}_s L - \lambda \mathcal{D}_s & -\mathcal{D}_s z \\ -(\mathcal{D}_s z)^* & 0 \end{pmatrix},$$

it is easy to check that

$$\text{Ker}(\text{Hess} \mathcal{A}_{H_0}(z_k, \bar{\lambda}_k)) = T_{(z_k, \bar{\lambda}_k)} \sigma_k.$$

Therefore, \mathcal{A}_{H_0} is a Morse-Bott functional. As done in the construction of the homology in subsection 5.2, for each $k \in \mathbb{Z}$ we can choose a Morse function h_k and a Riemannian metric g_k on σ_k such that h_k has precisely a maximum point p_k^+ and a minimum point p_k^- . It follows that

$$\nu(p_k^+) = i_{\text{rel}}(p_k^+) + 2m_k - 1 \quad \text{and} \quad \nu(p_k^-) = i_{\text{rel}}(p_k^-)$$

since $\text{ind}(p_k^+) = 2m_k - 1$ and $\text{ind}(p_k^-) = 0$. Let h be the Morse function on the critical manifold of \mathcal{A}_{H_0} , which coincides with h_k on each connected component σ_k . To compute the relative indices of critical points of \mathcal{A}_{H_0} , we need

Lemma 7.1 ([3, Proposition 2.4]). *Let H_1, H_2 be two Hilbert spaces, and $T : H_1 \rightarrow H_2$ an injective bounded linear operator. If U, V are commensurable subspaces of H_1 , then $T(U), T(V)$ are commensurable subspaces of H_2 and $\dim(T(U), T(V)) = \dim(U, V)$.*

Denote by

$$A = \mathcal{D}_s L \oplus Id_{\mathbb{R}}, \quad B(z, \lambda) = \text{Hess} \mathcal{A}_H(z, \lambda).$$

Consider unbounded self-adjoint operators on $L^2(M, \Sigma M) \times L^2(M, \Sigma M) \times \mathbb{R}$ defined by

$$C = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}, \quad D(z, \lambda) = \begin{pmatrix} L - \lambda Id & -z \\ -(z)^* & 0 \end{pmatrix}.$$

Let $j : \mathcal{E} \hookrightarrow L^2(M, \Sigma M) \times L^2(M, \Sigma M) \times \mathbb{R}$ be the inclusion map. Then (2.5) implies that

$$(Aw, w)_{\mathcal{E}} = (C(j(w)), j(w))_{L^2 \times \mathbb{R}} \quad \forall w \in \mathcal{E}, \quad (7.2)$$

$$(B(z, \lambda)w, w)_{\mathcal{E}} = (D(z, \lambda)(j(w)), j(w))_{L^2 \times \mathbb{R}} \quad \forall w \in \mathcal{E}. \quad (7.3)$$

$$i_{\text{rel}}(w) = \dim(V^-(D(w), V^-(C))).$$
$$-\infty \downarrow \bar{\lambda}_{-k} < \cdots < \bar{\lambda}_{-1} < 0 < \bar{\lambda}_1 < \cdots < \bar{\lambda}_k \uparrow +\infty, \quad k \in \mathbb{N}.$$

For each $l \in \mathbb{Z} \setminus \{0\}$, denote by $z_{l,1}, \dots, z_{l,m_l}$ the orthogonal eigenvectors of L with L^2 -norm $\sqrt{2}$ corresponding to $\bar{\lambda}_l$ with multiplicity m_l . Given $\bar{\lambda}_k$, $k \in \mathbb{Z}$, $n \in \{1, \dots, m_k\}$, let

$$z = \sum_{l \in \mathbb{Z} \setminus \{0\}} a_{l,1} z_{l,1} + \cdots + a_{l,m_l} z_{l,m_l}$$

with $a_{l,1}, \dots, a_{l,m_l} \in \mathbb{C}$, such that $D(z_{k,n}, \bar{\lambda}_k)(z, \mu)^t = \lambda(z, \mu)^t$. Then we have

$$-(z_{k,n}, z)_{L^2} = \lambda \mu \quad (7.4)$$

[illegible]

for each $l \in \mathbb{Z}$. It follows from (7.4) and (7.5) that either $\mu \neq 0$ and so

$$z = -\frac{\mu}{\lambda} z_{k,n} \quad \text{and} \quad \lambda = \pm \|z_{k,n}\|_{L^2},$$

or $\mu = 0$ and hence $z = a_{l,1}z_{l,1} + \cdots + a_{l,m_l}z_{l,m_l}$ for some $l \neq k$ and $\lambda = \bar{\lambda}_l - \bar{\lambda}_k$.

Summarizing up the above computation, we obtain that

$$V^-(C) = \bigoplus_{l \in \mathbb{N}} \text{span}_{\mathbb{C}}\{(z_{-l,1}, 0), \dots, (z_{-l,m_{-l}}, 0)\} \quad \text{and}$$

$$V^-(D(z_{k,n}, \bar{\lambda}_k)) = \bigoplus_{l < k} \text{span}_{\mathbb{C}}\{(z_{l,1}, 0), \dots, (z_{l,m_l}, 0)\} \bigoplus \text{span}_{\mathbb{R}}\{(\frac{\sqrt{2}}{2}z_{k,n}, 1)\}.$$

Let us discuss the relative index at $(z_{k,n}, \bar{\lambda}_k)$ in two cases.

Case 1. If $k < 0$, then

$$\begin{aligned} V^-(D(z_{k,n}, \bar{\lambda}_k)) \cap (V^-(C))^\perp &= \{0\} \quad \text{and} \\ [V^-(D(z_{k,n}, \bar{\lambda}_k))]^\perp \cap V^-(C) &= \bigoplus_{k \leq l < 0} \text{span}_{\mathbb{C}}\{(z_{-l,1}, 0), \dots, (z_{-l, m_{-l}}, 0)\}. \end{aligned}$$

It follows that

$$i_{\text{rel}}(z_{k,n}, \bar{\lambda}_k) = -2 \sum_{k \leq l < 0} m_l,$$

which implies

$$\nu(p_k^+) = -1 - 2 \sum_{k+1 \leq l < 0} m_l \quad \text{and} \quad \nu(p_k^-) = -2 \sum_{k \leq l < 0} m_l.$$

Case 2. If $k > 0$, then

$$V^-(D(z_{k,n}, \bar{\lambda}_k)) \cap (V^-(C))^\perp = \bigoplus_{0 < l < k} \text{span}_{\mathbb{C}}\{(z_{l,1}, 0), \dots, (z_{l, m_l}, 0)\} \bigoplus \text{span}_{\mathbb{R}}\{(\frac{\sqrt{2}}{2} z_{k,n}, 1)\}$$

and

$$(V^-(D(z_{k,n}, \bar{\lambda}_k)))^\perp \cap V^-(C) = \{0\}.$$

These lead to

$$i_{\text{rel}}(z_{k,n}, \bar{\lambda}_k) = 1 + 2 \sum_{0 < l < k} m_l,$$

and thus

$$\nu(p_k^+) = 2 \sum_{0 < l \leq k} m_l \quad \text{and} \quad \nu(p_k^-) = 1 + 2 \sum_{0 < l < k} m_l.$$

Since in both cases it holds that

$$\begin{aligned} \nu(p_k^-) - \nu(p_{k-1}^+) &= 1 \quad \text{for } k \neq 0, 1, \\ \nu(p_{\pm 1}^-) &= \pm 1 \quad \text{and} \quad \nu(p_1^-) - \nu(p_{-1}^+) = 2, \end{aligned}$$

we have one generator for each $\nu(p_k^\pm)$ in the chain complex $BC_*(H, K, g, h)$ for generic choice of K and g , that is to say,

$$BC_*(H_0, K, g, h) = \begin{cases} \mathbb{Z}_2, & * = \pm 1, \\ \mathbb{Z}_2, & * = 2 \sum_{1 \leq j \leq k} m_j, \quad k \in \mathbb{N}, \\ \mathbb{Z}_2, & * = 1 + 2 \sum_{1 \leq j \leq k} m_j, \quad k \in \mathbb{N}, \\ \mathbb{Z}_2, & * = -2 \sum_{1 \leq j \leq k} m_{-j}, \quad k \in \mathbb{N}, \\ \mathbb{Z}_2, & * = -1 - 2 \sum_{1 \leq j \leq k} m_{-j}, \quad k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \quad (7.6)$$

Notice that $\nu(p_k^+) - \nu(p_k^-) \geq 3$ for $m_k > 1$. But for $m_k = 1$ one can choose (h_k, g_k) such that there exist precisely two flow lines from p_k^+ to p_k^- on the connected component $\sigma_k \approx S^1$. In both cases, we get

$$\partial_{\nu(p_k^+)} = 0 \quad \text{for all } k \in \mathbb{Z} \setminus \{0\}.$$

To compute homology it remains to compute $\partial_{\nu(p_k^-)}$. Since there exists no generators of index 0, the image of ∂_0 must be zero. Then $\ker \partial_{-1} = \mathbb{Z}_2$ implies that

$$RHF_{-1}(H_0) = HF_{-1}(H_0) = \mathbb{Z}_2. \quad (7.7)$$

Furthermore, by continuation we have

Theorem 7.2. *Let $H \in C^2(\Sigma M \oplus \Sigma M)$ satisfy (H1) – (H4). Then*

$$RHF_{-1}(H) = RHF_{-1}(H_0) = \mathbb{Z}_2. \quad (7.8)$$

7.2 Proof of Theorem 1.1

By the transversality results in Section 6 and Theorem 7.2 there exist a sequence of smooth functions $H_n : \Sigma M \oplus \Sigma M \rightarrow \mathbb{R}$ satisfying (H1) – (H4) and the following conditions

- $H - H_n \in \mathbf{G} \forall n$, and $\varepsilon_n := \|H - H_n\|_{C^2} \rightarrow 0$ as $n \rightarrow \infty$;
- $RHF_{-1}^{[a,b]}(H_n) \neq 0 \forall n$ for some two real numbers $a < b$.

Thus \mathcal{A}_{H_n} has critical points (z_n, λ_n) satisfying

$$\mathcal{A}_{H_n}(z_n, \lambda_n) \in [a, b].$$

From the proof of Proposition 3.1 we see that $\|(z_n, \lambda_n)\|_{\mathcal{E}}$ is bounded by some constant C_1 which is independent of H_n . Then

$$\begin{aligned} |\mathcal{A}_H(z_n, \lambda_n) - \mathcal{A}_{H_n}(z_n, \lambda_n)| &\leq |\lambda_n| \int_M |H(x, z_n) - H_n(x, z_n)| \\ &\leq C_1 \varepsilon_n \end{aligned} \quad (7.9)$$

and

$$\begin{aligned} \|\nabla \mathcal{A}_H(z_n, \lambda_n) - \nabla \mathcal{A}_{H_n}(z_n, \lambda_n)\|_{\mathcal{E}} &\leq \left\| \mathcal{D}_s \left\{ \frac{\partial H}{\partial z}(x, z_n) - \frac{\partial H_n}{\partial z}(x, z_n) \right\} \right\| \\ &\quad + |\lambda_n| \int_M |H(x, z_n) - H_n(x, z_n)| \\ &\leq C_0 \varepsilon_n + C_1 \varepsilon_n, \end{aligned} \quad (7.10)$$

where in (7.10) we have use the facts: $D_s : E_s^* \rightarrow E_s$ is bounded and $L^\infty \times L^\infty \subset E_s^*$, which imply that the first term of the right side is bounded by $C_0 \|H_z - H_{nz}\|_{L^\infty} \leq C_0 \varepsilon_n$. From (7.9) we may assume that

$$(z_n, \lambda_n) \in \mathcal{A}_H^{-1}[a - K, b + K] =: \mathcal{S} \subset \mathcal{E}$$

for some constant $K > 0$. Let

$$\alpha := \inf_{(z, \lambda) \in \mathcal{S}} \nabla \mathcal{A}_H(z, \lambda). \quad (7.11)$$

We claim that $\alpha = 0$. By a contradiction, suppose $\alpha > 0$. Then (7.10) yields

$$\begin{aligned} \|\nabla \mathcal{A}_{H_n}(z_n, \lambda_n)\|_{\mathcal{E}} &\geq \|\nabla \mathcal{A}_H(z_n, \lambda_n)\|_{\mathcal{E}} - \|\nabla \mathcal{A}_H(z_n, \lambda_n) - \nabla \mathcal{A}_{H_n}(z_n, \lambda_n)\|_{\mathcal{E}} \\ &\geq \alpha - C_0 \varepsilon_n - C_1 \varepsilon_n > 0 \end{aligned}$$

for very large n . This contradiction shows $\alpha = 0$.

Now by (7.11) one can choose a sequence $\{(z_n, \lambda_n)\}_{n=1}^\infty \subset \mathcal{S} \subset \mathcal{E}$ such that

$$\begin{cases} \mathcal{A}_H(z_n, \lambda_n) \rightarrow c \in [a - K, b + K], \\ \nabla \mathcal{A}_H(z_n, \lambda_n) \rightarrow 0, \quad n \rightarrow \infty. \end{cases}$$

That is, $\{(z_n, \lambda_n)\}_{n=1}^\infty$ is a $(PS)_c$ -sequence. It follows from Proposition 3.1 that there exists a subsequence of $\{(z_n, \lambda_n)\}_{n=1}^\infty$ converging to (z^*, λ^*) in \mathcal{E} , which is a critical point of \mathcal{A}_H . Then (2.12) implies that $z^* = (u^*, v^*) \neq 0$ and $\lambda^* \neq 0$. Put

$$\begin{cases} u_0 = \lambda^* \frac{q+1}{1-pq} u^*, \\ v_0 = \lambda^* \frac{p+1}{1-pq} v^*. \end{cases}$$

Then (u_0, v_0) is a nontrivial weak solution of the Dirac system (1.13) and thus is of class C^1 by the elliptic regularity.

A Proof of Proposition 2.1

Step 1. \mathcal{H} is Gâteaux differentiable. Given $z = (u, v) \in E_s$, $h = (\xi, \zeta) \in E_s$ and $t \in (-1, 1) \setminus \{0\}$ we have, by the mean value theorem,

$$\begin{aligned} & \frac{H(x, u(x) + t\xi(x), v(x) + t\zeta(x)) - H(x, u(x), v(x))}{t} \\ &= \langle H_u(x, u(x) + \theta_1 \xi(x), v(x) + t\zeta(x)), \xi(x) \rangle + \langle H_v(x, u(x), v(x) + \theta_2 \zeta(x)), \zeta(x) \rangle \end{aligned}$$

for some $\theta_j = \theta(t, x, z(x), h(x)) \in (0, 1)$, $j = 1, 2$. It follows from the condition **(H2)** that

$$\begin{aligned} & \left| \frac{H(x, u(x) + t\xi(x), v(x) + t\zeta(x)) - H(x, u(x), v(x))}{t} \right| \\ & \leq c_1 \left(1 + |u(x) + \theta_1 \xi(x)|^p + |v(x) + t\zeta(x)|^{\frac{p(q+1)}{p+1}} \right) |\xi(x)| \\ & \quad + c_1 \left(1 + |u(x)|^{\frac{q(p+1)}{q+1}} + |v(x) + \theta_2 \zeta(x)|^q \right) |\zeta(x)| \\ & \leq c_1 \left(1 + 2^p |u(x)|^p + 2^p |\xi(x)|^p + 2^{\frac{p(q+1)}{p+1}} |v(x)|^{\frac{p(q+1)}{p+1}} + 2^{\frac{p(q+1)}{p+1}} |\zeta(x)|^{\frac{p(q+1)}{p+1}} \right) |\xi(x)| \\ & \quad + c_1 \left(1 + |u(x)|^{\frac{q(p+1)}{q+1}} + 2^q |v(x)| + 2^q |\zeta(x)|^q \right) |\zeta(x)|. \end{aligned}$$

Hereafter c_1, c_2, \dots , denote constants only depending on H and M . By the Höler inequality and the Sobolev embedding theorem it is easily checked that the last two lines of the above inequalities are integrable. Using the Lebesgue Dominated Convergence Theorem we deduce

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{\mathcal{H}(x, u + t\xi, v + t\zeta) - \mathcal{H}(x, u, v)}{t} \\ &= \int_M \lim_{t \rightarrow 0} \frac{H(x, u(x) + t\xi(x), v(x) + t\zeta(x)) - H(x, u(x), v(x))}{t} dx \\ &= \int_M \{ \langle H_u(x, u(x), v(x)), \xi(x) \rangle + \langle H_v(x, u(x), v(x)), \zeta(x) \rangle \} dx \end{aligned}$$

because $H \in C^2(\Sigma M \oplus \Sigma M)$. Notice that **(H2)** implies

$$\begin{aligned} \int_M |\langle H_u(x, u, v), \xi \rangle| dx &\leq c_1 \int_M (1 + |u|^p + |v|^{\frac{p(q+1)}{p+1}}) |\xi| dx, \\ \int_M |\langle H_v(x, u, v), \zeta \rangle| dx &\leq c_1 \int_M (1 + |u|^{\frac{q(p+1)}{q+1}} + |v|^q) |\zeta| dx. \end{aligned}$$

It follows from Höler inequalities, (1.5) and Sobolev embeddings that

$$\begin{aligned} \int_M |\langle H_u(x, u, v), \xi \rangle| dx &\leq C(1 + \|u\|_{s,2}^p + \|v\|_{1-s,2}^{\frac{p(q+1)}{p+1}}) \|\xi\|_s, \\ \int_M |\langle H_v(x, u, v), \zeta \rangle| dx &\leq C(1 + \|u\|_{s,2}^{\frac{q(p+1)}{q+1}} + \|v\|_{1-s,2}^q) \|\zeta\|_{1-s}. \end{aligned}$$

Hence the Gâteaux derivative $D\mathcal{H}(z) = D\mathcal{H}(u, v)$ exists and

$$\begin{aligned} D\mathcal{H}(z)h &= D\mathcal{H}(u, v)(\xi, \zeta) \\ &= \int_M \{ \langle H_u(x, u(x), v(x)), \xi(x) \rangle + \langle H_v(x, u(x), v(x)), \zeta(x) \rangle \} dx \\ &= \int_M H_z(x, z(x)) h(x) dx. \end{aligned} \tag{A.1}$$

Step 2. $D\mathcal{H} : E_s \rightarrow E_s^*$ is continuous and thus \mathcal{H} has continuous (Fréchet) derivative $\mathcal{H}' = D\mathcal{H}$. Consequently, the gradient $\nabla \mathcal{H}(z)$ at z is given by

$$\nabla \mathcal{H}(z) = \mathcal{D}_s H_z(\cdot, z). \tag{A.2}$$

Let

$$\hat{r}_1 = \frac{2n}{n-2s}, \quad \hat{r}_2 = \frac{2n}{n-2(1-s)}.$$

(H2) implies that if $r_1 \geq p+1$ and $r_2 \geq q+1$ then there exists a constant $c_3 > 0$ such that

$$|H_u(x, u, v)| \leq c_3(1 + |u|^{r_1-1} + |v|^{r_2(r_1-1)/r_1}), \tag{A.3}$$

$$|H_v(x, u, v)| \leq c_3(1 + |u|^{r_1(r_2-1)/r_2} + |v|^{r_2-1}). \tag{A.4}$$

Obviously, the above two inequalities yield

$$|H_u(x, u, v)|^{r_1/(r_1-1)} \leq c_4(1 + |u|^{r_1} + |v|^{r_2}), \tag{A.5}$$

$$|H_v(x, u, v)|^{r_2/(r_2-1)} \leq c_4(1 + |u|^{r_1} + |v|^{r_2}). \tag{A.6}$$

By the Sobolev embedding we can find two constants $C_i, i = 1, 2$, such that

$$\begin{aligned} \|u\|_{L^{2n/(n-2s)}} &\leq C_1 \|u\|_s \quad \forall u \in H^s(M, \Sigma M), \\ \|v\|_{L^{2n/(n-2(1-s))}} &\leq C_2 \|v\|_{1-s} \quad \forall v \in H^{1-s}(M, \Sigma M). \end{aligned}$$

In particular, there is a constant $a_{r_1, r_2} > 0$ such that for any $z = (u, v) \in E_s = H^s(M, \Sigma M) \times H^{1-s}(M, \Sigma M)$ we have

$$\|u\|_{L^{r_1}} + \|v\|_{L^{r_2}} \leq a_{r_1, r_2} \|z\|. \tag{A.7}$$

Given $z = (u, v), h = (h_1, h_2) \in E_s$, combining (A.3) and (A.5) gives

$$\begin{aligned}
& \int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_1/(r_1-1)} dx \\
& \leq c_5 \int_M (1 + |u|^{r_1} + |h_1|^{r_1} + |v|^{r_2} + |h_2|^{r_2}) dx \\
& \leq c_5 (1 + \|z\|^{r_1} + \|z\|^{r_2} + \|h\|^{r_1} + \|h\|^{r_2}).
\end{aligned} \tag{A.8}$$

Similarly, from (A.4) and (A.5) we arrive at

$$\begin{aligned}
& \int_M |H_v(x, z(x) + h(x)) - H_v(x, z(x))|^{r_2/(r_2-1)} dx \\
& \leq c_6 (1 + \|z\|^{r_1} + \|z\|^{r_2} + \|h\|^{r_1} + \|h\|^{r_2}).
\end{aligned} \tag{A.9}$$

By the definition and the Hölder inequality we get

$$\begin{aligned}
& \|D\mathcal{H}(z + h) - D\mathcal{H}(z)\|_{E_s^*} = \sup_{\|g\| \leq 1} |D\mathcal{H}(z + h) - D\mathcal{H}(z), g| \\
& = \sup_{\|g\| \leq 1} \left[\int_M (|H_u(x, z(x) + h(x)) - H_u(x, z(x))| |g_1(x)| \right. \\
& \quad \left. + |H_v(x, z(x) + h(x)) - H_v(x, z(x))| |g_2(x)|) dx \right] \\
& \leq \sup_{\|g\| \leq 1} \left[\|g_1\|_{L^{r_1}} \times \left(\int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_1/(r_1-1)} dx \right)^{(r_1-1)/r_1} \right. \\
& \quad \left. + \|g_2\|_{L^{r_2}} \times \left(\int_M |H_v(x, z(x) + h(x)) - H_v(x, z(x))|^{r_2/(r_2-1)} dx \right)^{(r_2-1)/r_2} \right] \\
& \leq a_{r_1, r_2} \left[\left(\int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_1/(r_1-1)} dx \right)^{(r_1-1)/r_1} \right. \\
& \quad \left. + \left(\int_M |H_v(x, z(x) + h(x)) - H_v(x, z(x))|^{r_2/(r_2-1)} dx \right)^{(r_2-1)/r_2} \right],
\end{aligned} \tag{A.10}$$

where we have used (A.5) in the last inequality. We deduce from (A.5) and (A.8) that the Nemytski map

$$N_{H_u} : L^{r_1}(M, \Sigma M) \times L^{r_2}(M, \Sigma M) \rightarrow L^{\frac{r_1}{r_1-1}}(M, \Sigma M), \quad h \mapsto H_u(\cdot, h(\cdot))$$

is continuous. Similarly, (A.6) and (A.9) imply the Nemytski map

$$N_{H_v} : L^{r_1}(M, \Sigma M) \times L^{r_2}(M, \Sigma M) \rightarrow L^{\frac{r_2}{r_2-1}}(M, \Sigma M), \quad h \mapsto H_v(\cdot, h(\cdot))$$

is also continuous. Then by the Sobolev embedding we have

$$\int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_1/(r_1-1)} dx \rightarrow 0, \tag{A.11}$$

$$\int_M |H_v(x, z(x) + h(x)) - H_v(x, z(x))|^{r_2/(r_2-1)} dx \rightarrow 0 \tag{A.12}$$

as $\|h\| \rightarrow 0$. Combining (A.10), (A.11) and (A.12) yields

$$\|D\mathcal{H}(z + h) - D\mathcal{H}(z)\|_{\mathcal{L}(E_s, E_s^*)} \rightarrow 0 \quad \text{as } \|h\| \rightarrow 0.$$

Step 3. \mathcal{H}' is a compact map. Suppose that $(z_k) \subset E_s$ is bounded. Passing to a subsequence, one may assume that z_k converges weakly in E_s to $z = (u, v)$. By the Rellich embedding theorem and the continuousness of $\mathcal{H}' = D\mathcal{H}$ we obtain that $\mathcal{H}'(z_k) \rightarrow \mathcal{H}'(z)$ as $k \rightarrow \infty$.

Step 4. \mathcal{H}' is of class C^1 . Given $x \in M$, $z = (u, v)$, $h = (h_1, h_2)$, $g = (g_1, g_2) \in E_s$, and $t \in (-1, 1) \setminus \{0\}$, the mean value theorem implies that

$$\begin{aligned} & \mathcal{H}'(z + th)g - \mathcal{H}'(z)g \\ &= \int_M (H_z(x, z(x) + th(x)) - H_z(x, z(x)))g(x)dx \\ &= \int_M \langle H_{zz}(x, z(x) + \theta th(x))th(x), g(x) \rangle dx \end{aligned} \quad (\text{A.13})$$

with $\theta = \theta(x, t, z(x), h(x)) \in (0, 1)$. Let $s_1 := \frac{n}{2s}$ and $s_2 := \frac{n}{2(1-s)}$. Then we have

$$\frac{2}{\hat{r}_1} + \frac{1}{s_1} = 1 \quad \text{and} \quad \frac{2}{\hat{r}_2} + \frac{1}{s_2} = 1. \quad (\text{A.14})$$

(H3) implies that there is a constant $a_1 > 0$ such that for any $z = (u, v) \in \Sigma_x M \oplus \Sigma_x M$ it holds that

$$\begin{aligned} |H_{uu}(x, z(x))|^{s_1} &\leq a_1(1 + |u(x)|^{p-1})^{s_1}, \\ |H_{vv}(x, z(x))|^{s_2} &\leq a_1(1 + |v(x)|^{q-1})^{s_2}, \\ |H_{uv}(x, z(x))|^n &\leq a_1, \quad |H_{vu}(x, z(x))|^n \leq a_1. \end{aligned} \quad (\text{A.15})$$

Notice that $0 < (p-1)s_1 < \hat{r}_1$ and $0 < (q-1)s_2 < \hat{r}_2$, using the Hölder inequality gives

$$\begin{aligned} & \int_M |H_{uu}(x, z(x)) + \theta th(x) - H_{uu}(x, z(x))|^{s_1} dx \\ &\leq a_2 \int_M (1 + |u(x)|^{s_1(p-1)} + |h_1(x)|^{s_1(p-1)}) dx \\ &\leq a_3(1 + \|u\|_{L^{\hat{r}_1}}^{s_1(p-1)} + \|h_1\|_{L^{\hat{r}_1}}^{s_1(p-1)}), \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} & \int_M |H_{vv}(x, z(x)) + \theta th(x) - H_{vv}(x, z(x))|^{s_2} dx \\ &\leq a_4 \int_M (1 + |v(x)|^{s_2(q-1)} + |h_2(x)|^{s_2(q-1)}) dx \\ &\leq a_5(1 + \|v\|_{L^{\hat{r}_2}}^{s_2(q-1)} + \|h_2\|_{L^{\hat{r}_2}}^{s_2(q-1)}), \end{aligned} \quad (\text{A.17})$$

$$\int_M |H_{uv}(x, z(x)) + \theta th(x) - H_{uv}(x, z(x))|^n dx \leq a_6 \quad (\text{A.18})$$

and

$$\int_M |H_{vu}(x, z(x)) + \theta th(x) - H_{vu}(x, z(x))|^n dx \leq a_6 \quad (\text{A.19})$$

It follows from (A.13) and the Hölder inequality that

$$\begin{aligned}
& \left\| \frac{1}{t} (\mathcal{H}'(z + th) - \mathcal{H}'(z)) - H_{zz}(\cdot, z)h \right\|_{E_s^*} \\
& \leq \sup_{\|g\| \leq 1} \left| \frac{1}{t} (\mathcal{H}'(z + th)g - \mathcal{H}'(z)g) - \langle H_{zz}(\cdot, z)h, g \rangle \right| \\
& \leq \sup_{\|g\| \leq 1} \left| \int_M \langle H_{zz}(x, z(x) + \theta th(x))h(x) - H_{zz}(x, z(x))h(x), g(x) \rangle dx \right| \\
& \leq \sup_{\|g\| \leq 1} \int_M [|H_{uu}(x, z(x) + \theta th(x)) - H_{uu}(x, z(x))| |h_1(x)| |g_1(x)| \\
& \quad + |H_{uv}(x, z(x) + \theta th(x)) - H_{uv}(x, z(x))| |h_2(x)| |g_1(x)| \\
& \quad + |H_{vu}(x, z(x) + \theta th(x)) - H_{vu}(x, z(x))| |h_1(x)| |g_2(x)| \\
& \quad + |H_{vv}(x, z(x) + \theta th(x)) - H_{vv}(x, z(x))| |h_2(x)| |g_2(x)|] dx \\
& \leq \sup_{\|g\| \leq 1} [\|H_{uu}(\cdot, z + \theta th) - H_{uu}(\cdot, z)\|_{L^{s_1}} \|h_1\|_{L^{\hat{r}_1}} \|g_1\|_{L^{\hat{r}_1}} \\
& \quad + \|H_{uv}(\cdot, z + \theta th) - H_{uv}(\cdot, z)\|_{L^n} \|h_2\|_{L^{\hat{r}_2}} \|g_1\|_{L^{\hat{r}_1}} \\
& \quad + \|H_{vu}(\cdot, z + \theta th) - H_{vu}(\cdot, z)\|_{L^n} \|h_1\|_{L^{\hat{r}_1}} \|g_2\|_{L^{\hat{r}_2}} \\
& \quad + \|H_{vv}(\cdot, z + \theta th) - H_{vv}(\cdot, z)\|_{L^{s_2}} \|h_2\|_{L^{\hat{r}_2}} \|g_2\|_{L^{\hat{r}_2}}].
\end{aligned}$$

Combining (A.16)-(A.19), the Sobolev inequality and the Lebesgue dominated convergence theorem implies that the right hand side of the last one of the above inequalities tends to 0 as $t \rightarrow 0$.

Therefore $\mathcal{H}' : E_s \rightarrow E_s^*$ is Gâteaux differentiable and

$$D\mathcal{H}'(z)h = H_{zz}(\cdot, z)h. \quad (\text{A.20})$$

For $z, h \in E_s$ it holds that

$$\begin{aligned}
& \|D\mathcal{H}'(z+h) - D\mathcal{H}'(z)\|_{\mathcal{L}(E_s, E_s^*)} = \sup_{\|g\| \leq 1} \|D\mathcal{H}'(z+h)g - D\mathcal{H}'(z)g\|_{E_s^*} \\
& \leq \sup_{\|g\| \leq 1} \sup_{\|\kappa\| \leq 1} |\langle D\mathcal{H}'(z+h)g - D\mathcal{H}'(z)g, \kappa \rangle| \\
& = \sup_{\|g\| \leq 1} \sup_{\|\kappa\| \leq 1} \left| \int_M \langle H_{zz}(x, z(x) + h(x))g(x) - H_{zz}(x, z(x))g(x), \kappa(x) \rangle dx \right| \\
& \leq \sup_{\|g\| \leq 1} \sup_{\|\kappa\| \leq 1} \int_M [|H_{uu}(x, z(x) + h(x)) - H_{uu}(x, z(x))| |g_1(x)| |\kappa_1(x)| \\
& \quad + |H_{uv}(x, z(x) + h(x)) - H_{uv}(x, z(x))| |g_2(x)| |\kappa_1(x)| \\
& \quad + |H_{vu}(x, z(x) + h(x)) - H_{vu}(x, z(x))| |g_1(x)| |\kappa_2(x)| \\
& \quad + |H_{vv}(x, z(x) + h(x)) - H_{vv}(x, z(x))| |g_2(x)| |\kappa_2(x)|] dx \\
& \leq \sup_{\|g\| \leq 1} \sup_{\|\kappa\| \leq 1} [\|H_{uu}(\cdot, z+h) - H_{uu}(\cdot, z)\|_{L^{s_1}} \|g_1\|_{L^{\hat{r}_1}} \|\kappa_1\|_{L^{\hat{r}_1}} \\
& \quad + \|H_{uv}(\cdot, z+h) - H_{uv}(\cdot, z)\|_{L^n} \|g_2\|_{L^{\hat{r}_2}} \|\kappa_1\|_{L^{\hat{r}_1}} \\
& \quad + \|H_{vu}(\cdot, z+h) - H_{vu}(\cdot, z)\|_{L^n} \|g_1\|_{L^{\hat{r}_1}} \|\kappa_2\|_{L^{\hat{r}_2}} \\
& \quad + \|H_{vv}(\cdot, z+h) - H_{vv}(\cdot, z)\|_{L^{s_2}} \|g_2\|_{L^{\hat{r}_2}} \|\kappa_2\|_{L^{\hat{r}_2}}] \\
& \leq a_{\hat{r}_1, \hat{r}_2}^2 [\|H_{uu}(\cdot, z+h) - H_{uu}(\cdot, z)\|_{L^{s_1}} + \|H_{uv}(\cdot, z+h) - H_{uv}(\cdot, z)\|_{L^n} \\
& \quad + \|H_{vu}(\cdot, z+h) - H_{vu}(\cdot, z)\|_{L^n} + \|H_{vv}(\cdot, z+h) - H_{vv}(\cdot, z)\|_{L^{s_2}}], \quad (\text{A.21})
\end{aligned}$$

where we have used (A.5) in the last inequality. Obviously, (A.15) implies that there is a constant $b_1 > 0$ such that

$$\begin{aligned}
|H_{uu}(x, z(x))|^{s_1} & \leq b_1 (1 + |u(x)|^{\hat{r}_1}), \quad |H_{vv}(x, z(x))|^{s_2} \leq b_1 (1 + |v(x)|^{\hat{r}_2}), \\
|H_{uv}(x, z(x))|^n & \leq b_1, \quad |H_{vu}(x, z(x))|^n \leq b_1. \quad (\text{A.22})
\end{aligned}$$

Then the above inequalities imply that the Nemytski maps

$$\begin{aligned}
N_{H_{uu}} & : L^{\hat{r}_1}(M, \Sigma M) \times L^{\hat{r}_2}(M, \Sigma M) \rightarrow L^{s_1}(M, \text{End}(\Sigma M)), \quad h \mapsto H_{uu}(\cdot, h(\cdot)), \\
N_{H_{vv}} & : L^{\hat{r}_1}(M, \Sigma M) \times L^{\hat{r}_2}(M, \Sigma M) \rightarrow L^{s_2}(M, \text{End}(\Sigma M)), \quad h \mapsto H_{vv}(\cdot, h(\cdot)), \\
N_{H_{uv}} & : L^{\hat{r}_1}(M, \Sigma M) \times L^{\hat{r}_2}(M, \Sigma M) \rightarrow L^n(M, \text{End}(\Sigma M)), \quad h \mapsto H_{uv}(\cdot, h(\cdot)) \quad \text{and} \\
N_{H_{vu}} & : L^{\hat{r}_1}(M, \Sigma M) \times L^{\hat{r}_2}(M, \Sigma M) \rightarrow L^n(M, \text{End}(\Sigma M)), \quad h \mapsto H_{vu}(\cdot, h(\cdot))
\end{aligned}$$

are all continuous. From this, (A.21) and the Sobolev embedding we deduce that

$$\|D\mathcal{H}'(z+h) - D\mathcal{H}'(z)\|_{\mathcal{L}(E_s, E_s^*)} \rightarrow 0$$

as $\|h\| \rightarrow 0$. The desired result is proved. \square

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